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# Different Methods of Analysis of Progressive Type-II Censoring from Burr Type-XII Model in Presence Competing Risk Data

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**Abstract:** In this paper we will study the maximum likelihood estimation (MLE) for the unknown parameters of the Burr Type-XII model when the data is progressively type-II censored in presence competing risk data, On the basis of this type of censoring, we will use the classical methods and the Bayesian method. We will compare the classic methods, which include the ML and parametric bootstrap methods with the Bayes method under squared error loss (SEL) function, we propose to apply Markov chain Monte Carlo (MCMC) technique to carry out a Bayesian estimation procedure as will as to construct the credible intervals. An example of real data is provided for illustration. Finally, all available methods were compared using Monte Carlo simulation.

**Keywords:** Burr type XII distribution; Progressive type-II censoring; Competing risks; Gibbs and Metropolis-Hasting samplers; credible intervals; Bootstrap.

## 1. Introduction

The effects competing risks plays an important role in various fields in order to quickly reach results for example prostate cancer. In statistical literature this is known as the analysis of competing risks model. A lifetime experiment with $k = 2$ different risk factors competing for the failure of the experimental units is considered. The data for such a "competing risks model" consist of the lifetime of the failed item and an indicator variable which denotes the cause of failure. For example, the competing risks for a prostate cancer patient may include prostate cancer itself, heart disease and (all) other causes. The effects of the other competing risks may play an important role in survival studies on slowly progressing diseases such as prostate cancer. In engineering applications, the causes or risks may signify either multiple modes of failure for a complex unit or multiple components or subsystems which comprise an entire system. Occurrence of a system failure is caused by the earliest onset of any of these component failures. In this respect, the framework is that of a system with components connected in a series. Several studies have been carried out under this assumption and the risks follow different lifetime distributions, namely the exponential, lognormal, gamma, Weibull, generalized exponential or exponentiated Weibull; see for example Moeschberger et al. (2008), Pascual (2010), Cramer and Schmiedt (2011), Sarhan et al. (2010), Sankaran and Ansa (2008), Sarhan (2007), Alwasel (2009), Kundu and Basu (2000) and Kundu and Sarhan (2006).

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Censoring occurs when exact lifetimes are known only for a portion of the individuals or units under study, while for the remainder of the lifetimes information on them is partial. There are several types of censored tests. The most common censoring schemes are Type-I (time) censoring, where the life testing experiment will be terminated at a prescribed time T, and Type-II (failure) censoring, where the life testing experiment will be terminated upon the r-th (r is pre-fixed) failure. However, the conventional type-I and type-II censoring schemes do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment. Because of this lack of flexibility, a more general censoring scheme called progressive type-II censoring has been introduced.

In this paper, we consider competing risk data under progressively type-II censoring. The censoring scheme is defined as follows: consider an experiment in which $n$ units are placed on a life testing experiment. At the time of the first failure (say $X_{1:m:n}$) has occurred, $R_1$ units are randomly removed from the remaining $n-1$ surviving units. Similarly, at the time of the second failure (say $X_{2:m:n}$) has occurred, $R_2$ units from the remaining $n-2-R_1$ units are randomly removed. The test continues until the $m$-th failure at which time (say $X_{m:m:n}$) has occurred, all the remaining $R_m = n - m - R_1 - R_2 - \cdots - R_{m-1}$ units are removed. The $R_i^{'}s$ are fixed prior to the study. We note that prior to the experiment in the progressive type-II censoring, an integer $m < n$ is determined and the progressive type-II censoring scheme $(R_1, R_2, \cdots, R_m)$ with $R_i > 0$ and $\sum_{i=1}^{m} R_i + m = n$ is specified. During the experiment, the $i$-th failure is observed and immediately after the failure, $R_i$ functioning items are randomly removed from the test. At the time of each failure, one or more surviving units may be removed from the study at random. The data from a progressively Type-II censored sample is as follows: $(x_{1:m:n}, \delta_1, R_1)$, $(x_{2:m:n}, \delta_2, R_2)$,$\cdots$,$(x_{m:m:n}, \delta_m, R_m)$, where $x_{1:m:n} < x_{2:m:n} < \cdots < x_{m:m:n}$ denote the $m$ observed failure times, $\delta_1$,$\delta_2$,$\cdots$,$\delta_m$ denote the causes of failure, and $R_1$,$R_2$,$\cdots$,$R_m$ denote the number of units removed from the test at the failure times $x_{1:m:n} < x_{2:m:n} < \cdots < x_{m:m:n}$. Readers may refer to Balakrishnan (2007) and Balakrishnan and Aggarwala (2000) for extensive reviews of the literature on progressive censoring. Recently, some work has been done on progressive Type-II censoring scheme, like, Nie and Gui (2019), Qin and Gui (2020) and Boulkeroua et al. (2022).

The rest of this paper is organized as follows. In Section 2, we describe the formulation of the model. Estimation of the parameters is given in Section 3. In this section, the ML estimators of the parameters $\alpha, \beta_1$ and $\beta_2$, approximate confidence intervals and bootstrap confidence intervals are presented. We cover Bayes estimates and construction of credible intervals using the MCMC techniques in Section 4. We provide some simulation results in order to give an assessment of the performance of the different estimation methods in Section 5. A real data example is presented in Section 6 for illustration. Finally we conclude the paper in Section 7.

# 2 Model assumptions

We assume that there are only two causes of failure. The model studied in the paper satisfies the following assumptions

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1. We put $n$ independent and identical unites on the life test. The test is terminated when $m \leq n$, $m$ is pre-specified, units failed. There are two independent causes of failure directed to each unit.

2. The lifetime of unit i is denoted as $X_i$, $i = 1, 2, \cdots, n$. The time at which the unit $i$ fails due to cause $j$ is $X_{ij}$, $j = 1, 2$. That is, $X_i = \min\{X_{i1}, X_{i2}\}$.

3. The distribution of the random variable $X_{ij}$ is Burr type XII distribution that was first introduced by Burr (1942) with shape parameters $\alpha$ and $\beta_j$, $j = 1, 2$ and $i = 1, 2, \cdots, n$. That is, the $(pdf)$ of $X_{ij}$, $j = 1, 2$, for each $i = 1, 2, \cdots, n$, is

$$f_j(x) = \alpha\beta_j x^{\alpha-1}(1 + x^{\alpha})^{-(\beta_j+1)}, \ x > 0, \ (\alpha > 0, \ \beta_j > 0). \tag{1}$$

$$F_j(x) = 1 - (1 + x^{\alpha})^{-\beta_j}, \ x > 0. \tag{2}$$

The corresponding reliability and failure rate functions of this distribution at some $t$, are given, respectively by

$$S_j(t) = (1 + t^{\alpha})^{-\beta_j}, \ t > 0, \tag{3}$$

$$H_j(t) = \alpha\beta_j t^{\alpha-1}(1 + t^{\alpha})^{-1}, \ t > 0. \tag{4}$$

The two-parameter Burr type XII distribution has unimodal or decreasing failure rate function Equation (4). It is clear that the parameter $\beta_j$ does not affect the shape of failure rate function $H_j(t)$ and $\alpha$ is the shape parameter. Also, $H_j(t)$ has a unimodal curve when $\alpha > 1$, achieving a maximum at $x = \frac{(\alpha-1)^{1/\alpha}}{\alpha}$, and it has decreasing failure rate function when $\alpha \leq 1$. Thus the shape parametrs $\beta_j$ plays an important role for the distribution. Its capacity to assume various shapes often permits a good fit when used to describe biological, clinical or other experimental data.

4. When the first failure occurs: (1) we observe two quantities $X_{1:m:n}$ and $\delta_1 \in \{1, 2\}$; (2) $R_1$ of surviving unites are randomly selected and removed. When the $i-th$ failure occurs, $i = 2, 3, \cdots, m-1$: (1) we observe two quantities $X_{i:m:n}$ and $\delta_i \in \{1, 2\}$; (2) $R_i$ of surviving unites are randomly selected and removed. This experiment terminates when the $m-th$ failure occurs. When the $m-th$ failure occurs: (1) we observe the two quantities $X_{m:m:n}$ and $\delta_m \in \{1, 2\}$; (2) the rest $R_m = n - m - \sum_{i=1}^{m-1} R_i$ surviving units are all removed from the test.

Based on the above assumptions, the available data is a progressively type-II censored sample which contains the following: $(X_{1:m:n}, \delta_1, R_1)$, $(X_{2:m:n}, \delta_2, R_2)$,$\cdots$,$(X_{m:m:n}, \delta_m, R_m)$, where $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ denote the $m$ observed failure times, $\delta_1$, $\delta_2$,$\cdots$, $\delta_m$ denote the causes of failures, and $R_1$, $R_2$,$\cdots$,$R_m$ denote the number of units removed from the test at the failure times $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$. To simplify the notation we will use henceforth $X_i$ instead of $X_{i:m:n}$.

# 3 Estimation of the Parameters

In this section, we first estimate the parameters $\alpha$ and $\beta_j$, by considering the maximum likelihood (ML) methods, and then we compute the observed Fisher information based on the likelihood equations. These will enable us to develop pivotal quantities based on the limiting normal distribution, the resulting pivotal quantities can be used to develop approximate confidence interval for the parameters. Finally, using the ML estimates, we construct the parametric bootstrap confidence intervals.

## 3.1 Maximum Likelihood Estimation

Based on the observed sample $(X_{1:m:n}, \delta_1, R_1)$, $(X_{2:m:n}, \delta_2, R_2)$,$\cdots$,$(X_{m:m:n}, \delta_m, R_m)$, the likelihood function is;

$$\ell(\underline{x}; \alpha, \beta_1, \beta_2) \propto \alpha^m \beta_1^{m_1} \beta_2^{m_2} v(\alpha; \underline{x}) \exp[-(\beta_1 + \beta_2) \sum_{i=1}^{m} (R_i + 1) \log(1 + x_i^{\alpha})], \tag{5}$$

where

$$v(\alpha; \underline{x}) = \prod_{i=1}^{m} \frac{x_i^{\alpha-1}}{(1 + x_i^{\alpha})}. \tag{6}$$

The log-likelihood function without the additive constant can be written as follows;

$$\begin{aligned} L(\underline{x}; \alpha, \beta_1, \beta_2) \;\; = \;\; & m \log \alpha + m_1 \log \beta_1 + m_2 \log \beta_2 + (\alpha - 1) \sum_{i=1}^{m} \log(x_i) \\ & - \sum_{i=1}^{m} \log(1 + x_i^{\alpha}) - (\beta_1 + \beta_2) \sum_{i=1}^{m} (R_i + 1) \log(1 + x_i^{\alpha}). \end{aligned} \tag{7}$$

Upon differentiating (7) with respect to $\alpha$, $\beta_1$ and $\beta_2$, and equating each result to zero, three equations must be simultaneously satisfied to obtain MLEs of the parameters $\alpha$, $\beta_1$ and $\beta_2$. Then, we have

$$\frac{\partial L(\underline{x}; \alpha, \beta_1, \beta_2)}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m} \log(x_i) - \sum_{i=1}^{m} \frac{x_i^{\alpha} \log(x_i)}{(1 + x_i^{\alpha})} - (\beta_1 + \beta_2) \sum_{i=1}^{m} \frac{(R_i + 1) x_i^{\alpha} \log(x_i)}{(1 + x_i^{\alpha})}, \tag{8}$$

$$\frac{\partial L(\underline{x}; \alpha, \beta_1, \beta_2)}{\partial \beta_1} = \frac{m_1}{\beta_1} - \sum_{i=1}^{m} (R_i + 1) \log(1 + x_i^{\alpha}), \tag{9}$$

and

$$\frac{\partial L(\underline{x}; \alpha, \beta_1, \beta_2)}{\partial \beta_2} = \frac{m_2}{\beta_2} - \sum_{i=1}^{m} (R_i + 1) \log(1 + x_i^{\alpha}). \tag{10}$$

The MLEs of $\beta_1$ and $\beta_2$, respectively, as

$$\hat{\beta}_1(\alpha) = \frac{m_1}{\sum_{i=1}^{m} (R_i + 1) \log(1 + x_i^{\alpha})}, \tag{11}$$

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$$\hat{\beta}_2(\alpha) = \frac{m_2}{\sum_{i=1}^{m}(R_i+1)\log(1+x_i^{\alpha})}. \tag{12}$$

By substituting (11) and (12) in (8), we get

$$\frac{m}{\alpha} + \sum_{i=1}^{m}\log(x_i) - \sum_{i=1}^{m}\frac{x_i^{\alpha}\log(x_i)}{(1+x_i^{\alpha})} - \frac{m}{\sum_{i=1}^{m}(R_i+1)\log(1+x_i^{\alpha})}\sum_{i=1}^{m}\frac{(R_i+1)x_i^{\alpha}\log(x_i)}{(1+x_i^{\alpha})} = 0. \tag{13}$$

Thus, the MLE $\hat{\alpha}$ of the parameter $\alpha$ can be obtained by solving the nonlinear likelihood Equation (13) using, for example, the Newton-Raphson iteration scheme. The corresponding MLE $\hat{\beta}_1$ and $\hat{\beta}_2$ of the parameters $\beta_1$ and $\beta_2$ are computed from Equations. (11) and (12). To obtain a starting value for the root finding method, we can use the graphical method presented by Balakrishnan and Kateri (2008). We obtain the profile log-likelihood of $\alpha$ by substituting back $\hat{\beta}_1(\alpha)$ and $\hat{\beta}_2(\alpha)$ in (7) and ignoring the additive constant we obtain the profile log-likelihood of $\alpha$ as

$$p(\alpha) = m\log\alpha - m\log\left[\sum_{i=1}^{m}(R_i+1)\log(1+x_m^{\alpha})\right] + \alpha\sum_{i=1}^{m}\log(x_i) - \sum_{i=1}^{m}\log(1+x_i^{\alpha}), \tag{14}$$

and the MLE of $\alpha$ can be obtained by maximizing (13) with respect to $\alpha$.

## 3.2 Approximate confidence intervals

In this subsection, we derive the asymptotic distribution of MLEs to construct approximate confidence intervals for unknown parameters $\alpha$, $\beta_1$ and $\beta_2$. The asymptotic variances and covariances of the MLEs for parameters are given by elements of the inverse of the Fisher information matrix define as

$$\mathbf{I}_{ij} = E\left[-\frac{\partial^2 L}{\partial\theta_i\partial\theta_j}\right];\ \theta_1=\alpha,\ \theta_2=\beta_1 \text{ and } \theta_3=\beta_2 \text{ for } i,\ j=1,2,3. \tag{15}$$

In order to obtain an approximate confidence interval, the Fisher information matrix is replaced by its estimate, the observed information

$$\begin{bmatrix} -\frac{\partial^2 L}{\partial\alpha^2} & -\frac{\partial^2 L}{\partial\alpha\partial\beta_1} & -\frac{\partial^2 L}{\partial\alpha\partial\beta_2} \\ -\frac{\partial^2 L}{\partial\beta_1\partial\alpha} & -\frac{\partial^2 L}{\partial\beta_1^2} & -\frac{\partial^2 L}{\partial\beta_1\partial\beta_2} \\ -\frac{\partial^2 L}{\partial\beta_2\partial\alpha} & -\frac{\partial^2 L}{\partial\beta_2\partial\beta_1} & -\frac{\partial^2 L}{\partial\beta_2^2} \end{bmatrix}^{-1}_{(\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2)} = \begin{bmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}_1) & cov(\hat{\alpha},\hat{\beta}_2) \\ cov(\hat{\beta}_1,\hat{\alpha}) & var(\hat{\beta}_1) & cov(\hat{\beta}_1,\hat{\beta}_2) \\ cov(\hat{\beta}_2,\hat{\alpha}) & cov(\hat{\beta}_2,\hat{\beta}_1) & var(\hat{\beta}_2) \end{bmatrix},$$

with

$$\frac{\partial^2 L}{\partial\alpha^2} = -\frac{m}{\alpha^2} - \frac{x_i^{\alpha}\log^2(x_i)}{(1+x_i^{\alpha})^2} - k(\beta_1+\beta_2)\sum_{i=1}^{m}\frac{(R_i+1)x_i^{\alpha}\log^2(x_i)}{(1+x_i^{\alpha})^2}, \tag{16}$$

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$$\frac{\partial^2 L}{\partial\alpha\partial\beta_1} = \frac{\partial^2 L}{\partial\beta_1\partial\alpha} = -k(\beta_1+\beta_2)\sum_{i=1}^{m}\frac{(R_i+1)x_i^\alpha\log(x_i)}{(1+x_i^\alpha)^2}, \tag{17}$$

$$\frac{\partial^2 L}{\partial\alpha\partial\beta_2} = \frac{\partial^2 L}{\partial\beta_2\partial\alpha} = -k(\beta_1+\beta_2)\sum_{i=1}^{m}\frac{(R_i+1)x_i^\alpha\log(x_i)}{(1+x_i^\alpha)^2}, \tag{18}$$

$$\frac{\partial^2 L}{\partial\beta_1\partial\beta_2} = \frac{\partial^2 L}{\partial\beta_2\partial\beta_1} = 0, \tag{19}$$

$$\frac{\partial^2 L}{\partial\beta_1^2} = -\frac{m_1}{\beta_1^2}, \quad \frac{\partial^2 L}{\partial\beta_2^2} = -\frac{m_1}{\beta_2^2}. \tag{20}$$

The asymptotic normality of the MLE can be used to compute the approximate confidence intervals for parameters $\alpha$, $\beta_1$ and $\beta_2$. Therefore, $(1-\gamma)100\%$ confidence intervals for parameters $\alpha$, $\beta_1$ and $\beta_2$ become

$$\left(\hat{\alpha} \pm Z_{\gamma/2}\sqrt{var(\hat{\alpha})}\right), \ \left(\hat{\beta}_1 \pm Z_{\gamma/2}\sqrt{var(\hat{\beta}_1)}\right) \text{ and } \left(\hat{\beta}_2 \pm Z_{\gamma/2}\sqrt{var(\hat{\beta}_2)}\right), \tag{21}$$

where $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$.

## 3.3 Bootstrap confidence intervals

In this subsection, we propose to use two confidence intervals based on the parametric bootstrap methods: (i) percentile bootstrap method (Boot-p) based on the idea of Efron (1982). (ii) bootstrap-t method (Boot-t) based on the idea of Hall (1988). The confidence intervals of $R$ using both methods are illustrated briefly in the following steps:

**Step 1** From the original data $\underline{x} \equiv x_{1:m:n}, x_{2:m:n}, \cdots, x_{m:m:n}$ compute the MLE's of the parameters: say $\hat{\alpha}, \hat{\beta}_1$ and $\hat{\beta}_2$ by solving the equations (11-13).

**Step 2** Use $\hat{\alpha}, \hat{\beta}_1$ and $\hat{\beta}_2$ in Step 1 to generate a bootstrap sample $\underline{x}^*$ with the same values of $R_i, m$; and $n$ , $(i = 1, 2, .., m)$ using algorithm presented in Balakrishnan and Sandhu (1995). For each data point, we assigned the cause of failure as 1 or 2 with probability $(\beta_1/(\beta_1+\beta_2))$ and $(\beta_2/(\beta_1+\beta_2))$, respectively..

**Step 4** Repeat Steps 2 $B$ times representing $\left(\hat{\varphi}_l^{*(1)}, \hat{\varphi}_l^{*(2)}, \ \cdots, \ \hat{\varphi}_l^{*(B)}\right), l = 1, 2, 3.$ Where $\hat{\varphi}_1^* = \hat{\alpha}^*, \hat{\varphi}_2^* = \hat{\beta}_1^*$ and $\hat{\varphi}_3^* = \hat{\beta}_2^*$.

**Step 5** Arrange $\left(\hat{\varphi}_l^{*(1)}, \hat{\varphi}_l^{*(2)}, \ \cdots, \ \hat{\varphi}_l^{*(B)}\right)$ in an ascending order to obtain the bootstrap sample $\left(\hat{\varphi}_{l(1)}^*, \hat{\varphi}_{l(2)}^*, \ \cdots, \ \hat{\varphi}_{l(B)}^*\right)$.

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**I- Percentile bootstrap method (Boot-p)**

Let $G(x) = P(\hat{\varphi}_l^* \leq x)$ be the (*cdf*) of $\hat{\varphi}_l^*$. Define $\varphi_{lboot-p} = G^{-1}(x)$ for given $x$. The approximate bootstrap $100(1-\gamma)\%$ confidence interval of $\varphi_l$ are given by

$$\left[\varphi_{lBoot-p}(\frac{\gamma}{2}), \varphi_{lBoot-p}(1-\frac{\gamma}{2})\right].$$

**II- Bootstrap-t method (Boot-t)**

Compute the following statistic:

$$T_l^* = \frac{\sqrt{m}(\hat{\varphi}_l^* - \hat{\varphi}_l)}{\sqrt{Var(\hat{\varphi}_l^*)}},\ l = 1, 2, 3,$$

where $Var(\hat{\varphi}_l^*)$ are obtained using the Fisher information matrix. Using $T_l^*$ values, determine the upper and lower bounds of the $100(1-\gamma)\%$ confidence interval of $\varphi_l$ as follows: let $H(x) = P(T_l^* \leq x), l = 1, 2, 3$ be the (*cdf*) of $T_l^*$. For a given $x$, define

$$\hat{\varphi}_{lBoot-t}(x) = \hat{\varphi}_l + m^{-1/2}\sqrt{Var(\hat{\varphi}_l)}H^{-1}(x), \text{ for } l = 1, 2, 3.$$

Here also, $Var(\hat{\varphi}_l)$ can be computed as same as computing the $Var(\hat{\varphi}_l^*)$. The approximate $100(1-\gamma)\%$ confidence interval of $\varphi_l$ are given by

$$\left(\hat{\varphi}_{lBoot-t}(\frac{\gamma}{2}), \hat{\varphi}_{lBoot-t}(1-\frac{\gamma}{2})\right), \text{ for } l = 1, 2, 3.$$

Hall (1988) showed that the Boot-t confidence interval is better than the Boot-p confidence interval from an asymptotic point of view.

# 4 Bayes Estimation of Parameters using MCMC

This section describes Bayesian MCMC methods that have been used to estimate the parameters $\alpha$, $\beta_1$ and $\beta_2$ based on progressively type-II censored in presence competing risks from the Burr-XII distribution. The Bayesian approach is introduced and its computational implementation with MCMC algorithms is described. Gibbs sampling procedure and the Metropolis-Hastings (M-H) method are used to generate samples from the posterior density function and in turn compute the Bayes point estimates and also construct the corresponding credible intervals based on the generated posterior samples. For an exhaustive list of references and further details on the MCMC technique, the readers are referred to the monograph by Robert and Casella (2004), Rezaei et al. (2010), Kundu (2008) and Smith and Roberrs (1993). For computing the Bayes estimates, we assume mainly a squared error loss (SEL) function only; however, any other loss function can be easily incorporated.

In some situations where we do not have sufficient prior information, we can use non-informative prior distribution. This is particularly true for our study. For example, the non-informative uniform prior distribution can be used for parameters $\alpha$, $\beta_1$ and $\beta_2$. The joint posterior density will then be in proportion to the likelihood function.

Here we consider the more important case, we assume the following independent prior densities for $\alpha, \beta_1$ and $\beta_2$

$$\alpha \sim Gamma(a_1, b_1),\ \beta_1 \sim Gamma(a_2, b_2),\ \text{and}\ \ \beta_2 \sim Gamma(a_3, b_3), \tag{22}$$

where $(a_i, b_i), i = 1, 2, 3$ are known positive parameters. Noninformative priors follows by considering $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$. Interestingly, in this case the Bayes estimators coincide with the corresponding MLEs.
The expression for the posterior distribution can be obtained up to proportionality by multiplying the likelihood with the prior and this can be written as

$$\begin{aligned}\pi^*_{\alpha,\beta_1,\beta_2}(\alpha, \beta_1, \beta_2|\underline{x}) \quad \propto \quad & \alpha^{m+a_1-1}\beta_1^{m_1+a_2-1}\beta_2^{m_2+a_3-1}v(\alpha;\underline{x})\exp[-b_1\alpha]\exp[-\beta_1 b_2]\exp[-\beta_2 b_3] \\ & \times \exp[-(\beta_1+\beta_2)\sum_{i=1}^{m}(R_i+1)\log(1+x_i^{\alpha})],\end{aligned} \tag{23}$$

where $v(\alpha; \underline{x})$ is defined in (6).
The posterior is obviously complicated and no closed form inferences appear possible. We, therefore, propose to consider MCMC methods, namely the Gibbs sampler, to simulate samples from the posterior so that sample-based inferences can be easily drawn. From (23), the full conditional posterior distributions required to implement the MCMC sampler are given by

$$\pi^*_{\alpha}(\alpha|\beta_1, \beta_2, \underline{x}) \propto \alpha^{m+a_1-1}v(\alpha;\underline{x})\exp[-b_1\alpha]\exp[-(\beta_1+\beta_2)\sum_{i=1}^{m}(R_i+1)\log(1+x_i^{\alpha})], \tag{24}$$

$$\pi^*_{\beta_1}(\beta_1|\alpha, \beta_2, \underline{x}) \sim Gamma(m_1 + a_2, b_2 + \sum_{i=1}^{m}(R_i+1)\log(1+x_i^{\alpha})), \tag{25}$$

and

$$\pi^*_{\beta_2}(\beta_2|\alpha, \beta_1, \underline{x}) \sim Gamma(m_2 + a_3, b_3 + \sum_{i=1}^{m}(R_i+1)\log(1+x_i^{\alpha})). \tag{26}$$

From Equations (25) and (26), it can be seen that samples of $\beta_1$ and $\beta_2$ can be easily generated using any gamma generating routine. However, in our case, the conditional posterior distribution of $\alpha$ Equation (24) cannot be reduced analytically to well known distributions and therefore it is not possible to sample directly by standard methods, but the plot of it show that it is similar to normal distribution. So to generate random numbers from this distribution, we need to use the M-H method with normal proposal distribution. Thus, we proceed as follows:

**Step 1:** Start with an $(\alpha^{(0)})$ and set $t = 1$.

**Step 2:** Generate $\beta_1^{(t)}$ from $Gamma(m_1 + a_2, b_2 + \sum_{i=1}^{m}(R_i+1)\log(1+x_i^{\alpha^{(t-1)}}))$.

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**Step 3:** Generate $\beta_2^{(t)}$ from $Gamma(m_2 + a_3, b_3 + \sum_{i=1}^{m}(R_i + 1)\log(1 + x_i^{\alpha^{(t-1)}}))$.

**Step 4:** Using Metropolis-Hastings (see, Metropolis et al. (1953)), generate $\alpha^{(t)}$ from $\pi_\alpha^*(\alpha|\beta_1^{(t)}, \beta_2^{(t)}, \underline{x})$ with the $N(\alpha^{(t-1)}, \sigma^2)$ proposal distribution. Where $\sigma^2$ is from variance covariance matrix.

**Step 5:** Compute $\alpha^{(t)}, \beta_1^{(t)}$ and $\beta_2^{(t)}$.

**Step 6:** Set $t = t + 1$.

**Step 7:** Repeat Steps $2 - 5$ N times.

**Step 8:** We obtain the Bayes MCMC point estimate of $\theta \equiv (\alpha, \beta_1, \beta_2)$ as

$$E(\theta|data) = \frac{1}{N - M}\sum_{i=M+1}^{N}\theta^{(i)},$$

where $M$ is the burn-in period (that is, a number of iterations before the stationary distribution is achieved).

**Step 9:** To compute the credible intervals of $\theta$, usually, we take the quantiles of the sample as the endpoints of the interval. Order $\theta^{(M+1)}$, $\theta^{(M+2)}$,. . . ,$\theta^{(N)}$ as $\theta_{(1)}$, $\theta_{(2)}$,$\cdots$,$\theta_{(N-M)}$. Then the $100(1-\gamma)\%$ symmetric credible interval is

$$\left(R_{(\gamma/2(N-M))}, R_{((1-\gamma/2)(N-M))}\right). \tag{27}$$

# 5 Monte Carlo simulations

Here in these calculations, we primarily perform some simulation experiments to observe the behavior of the different methods. Monte Carlo simulations were performed utilizing 1000 progressively Type-II censored samples for each simulations. The samples were generated by using the algorithm described in Balakrishnan and Sandh (1995) using $(\alpha, \beta_1, \beta_2) = (2, 0.6, 0.4)$, different sample size $n$ and different sample size $m$, For each data point, we assigned the cause of failure as 1 or 2 with probability $(\beta_1/(\beta_1 + \beta_2))$ and $(\beta_2/(\beta_1 + \beta_2))$, respectively. A simulation was conducted in order to study the properties and compare the performance of different methods, namely the maximum likelihood estimator, bootstraps and Bayes estimator. We consider the following different sampling schemes:

$S_{m:n}^{(1)}$ : The first observation removals, i.e. $R_1 = n - m$, $R_i = 0$ for $i \neq 1$.
$S_{m:n}^{(2)}$ : The middle observation removal, i.e. $R_{\frac{m}{2}} = n - m$, $R_i = 0$ for $i \neq \frac{m}{2}$.
$S_{m:n}^{(3)}$ : The last observation removals, i.e. $R_m = n - m$, $R_i = 0$ for $i \neq m$.

*Curr. Sci. Int., 12(2): 102-116, 2023*
*EISSN: 2706-7920 ISSN: 2077-4435* **DOI: 10.36632/csi/2023.12.2.11**

The MLE $\hat{\alpha}$ of the parameter $\alpha$ is computed from the solution of Equation (13) using Newton-Raphson iteration. Once we estimate $\alpha$, we derived $\hat{\beta}_1$ and $\hat{\beta}_2$ using Equations (11) and (12), respectively.

We consider informative prior for the unknown parameters namely as ($a_1 = 2$, $b_1 = 1$, $a_2 = 3, b_2 = 5$, $a_3 = 2$, $b_3 = 5$) for the parameter values. We have chosen the hyper-parameters in such a way that the prior mean became the expected value of the corresponding parameter. We compute the average maximum likelihood estimates (AMEs), mean squared errors (MSEs), average 95% confidence interval lengths (ACILs) and the corresponding coverage percentages (CPs) of the parameters. In addition to the same previous computations for Boot_p and Boot_T based on 1000 bootstrap replications,. Also, We compute the average Bayes estimates (ABEs) with respect to squared error loss function, mean squared errors (MSEs), average 95% credible interval lengths (ACILs) and the corresponding coverage percentages (CPs) of the parameters based on 10000 MCMC samples and discard the first 1000 values as "burn-in". The results are reported in Tables 2 and 3. Based on the results of the simulation study some of the points are clear from this experiment. Even for some small sample sizes, we observe the following:

**(i)** From Tables 2 , as expected for all the methods, when $n$, $m$ increase then the AMEs and the MSEs decrease.

**(ii)** From Tables 2 and 3, it can be seen that the performance of the MLEs is quite close to that of the Boot-p and Boot-t methods.

**(iii)** From Tables 3, in most cases the estimated coverage probability is close to the nominal level of 0.95 based on different methods.

**(iv)** Also it can be seen that, in most cases the (MSEs) of the Bayes estimators perform much better than the MLEs, Boot-p and Boot-t methods.

*Curr. Sci. Int., 12(2): 102-116, 2023*
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Table 2: The AMEs, of parameters and their MSEs for different censoring schemes are reported, when $\alpha = 2$, $\beta_1 = 0.6$ and $\beta_2 = 0.4$.

| Scheme | MLE | | | Boot_p | | | Boot_T | | | Bayes MCMC | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\alpha$ | $\beta_1$ | $\beta_2$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\alpha$ | $\beta_1$ | $\beta_2$ |
| $S^{(1)}_{20:50}$ | 2.0727 | 0.6436 | 0.4237 | 2.1831 | 0.6908 | 0.4575 | 2.1321 | 0.6411 | 0.4129 | 2.0458 | 0.6221 | 0.4128 |
| | 0.0810 | 0.0469 | 0.0178 | 0.1137 | 0.0581 | 0.0254 | 0.0938 | 0.0488 | 0.0193 | 0.0694 | 0.0255 | 0.0097 |
| $S^{(2)}_{20:50}$ | 2.1474 | 0.6438 | 0.3812 | 2.2294 | 0.6759 | 0.4016 | 2.1645 | 0.6309 | 0.3589 | 2.1422 | 0.6257 | 0.3807 |
| | 0.1852 | 0.0485 | 0.0230 | 0.1971 | 0.0553 | 0.0259 | 0.1880 | 0.0485 | 0.0245 | 0.1497 | 0.0261 | 0.0134 |
| $S^{(3)}_{20:50}$ | 2.1937 | 0.7180 | 0.4612 | 2.3317 | 0.7572 | 0.4976 | 2.2764 | 0.7099 | 0.4512 | 2.1111 | 0.6497 | 0.4229 |
| | 0.2526 | 0.1481 | 0.0625 | 0.3217 | 0.1093 | 0.0591 | 0.3031 | 0.1368 | 0.0559 | 0.1602 | 0.0423 | 0.0178 |
| $S^{(1)}_{40:50}$ | 2.1065 | 0.6012 | 0.3938 | 2.1832 | 0.6111 | 0.3992 | 2.1251 | 0.5904 | 0.3809 | 2.1045 | 0.5993 | 0.3934 |
| | 0.0605 | 0.0114 | 0.0137 | 0.1322 | 0.0296 | 0.0135 | 0.0662 | 0.0219 | 0.0126 | 0.0571 | 0.0140 | 0.0101 |
| $S^{(2)}_{40:50}$ | 2.0505 | 0.5929 | 0.4107 | 2.1206 | 0.6023 | 0.4176 | 2.0814 | 0.5826 | 0.3998 | 2.0419 | 0.5933 | 0.4082 |
| | 0.0701 | 0.0122 | 0.0314 | 0.0907 | 0.0153 | 0.0148 | 0.0686 | 0.0121 | 0.0131 | 0.0692 | 0.0156 | 0.0189 |
| $S^{(3)}_{40:50}$ | 2.0217 | 0.6182 | 0.3931 | 2.0973 | 0.6297 | 0.4016 | 2.0511 | 0.6099 | 0.3832 | 2.0160 | 0.6143 | 0.3927 |
| | 0.0754 | 0.0221 | 0.0414 | 0.0801 | 0.0147 | 0.0152 | 0.0783 | 0.0119 | 0.0140 | 0.0802 | 0.0166 | 0.0087 |
| $S^{(1)}_{35:70}$ | 2.1163 | 0.6195 | 0.4345 | 2.1799 | 0.6365 | 0.4447 | 2.0426 | 0.6114 | 0.4213 | 2.1185 | 0.6125 | 0.4277 |
| | 0.1250 | 0.0244 | 0.0117 | 0.0856 | 0.0314 | 0.0143 | 0.0790 | 0.0276 | 0.0128 | 0.0648 | 0.0169 | 0.0094 |
| $S^{(2)}_{35:70}$ | 2.0152 | 0.6495 | 0.4602 | 2.0834 | 0.6757 | 0.4780 | 2.1468 | 0.6456 | 0.4523 | 2.0032 | 0.6358 | 0.4470 |
| | 0.0731 | 0.0287 | 0.0219 | 0.1513 | 0.0318 | 0.0147 | 0.1309 | 0.0249 | 0.0129 | 0.1144 | 0.0193 | 0.0133 |
| $S^{(3)}_{35:70}$ | 2.0715 | 0.6248 | 0.4019 | 2.1689 | 0.6577 | 0.4230 | 2.1235 | 0.6218 | 0.3962 | 2.0582 | 0.6167 | 0.3992 |
| | 0.1350 | 0.0294 | 0.0223 | 0.1684 | 0.0351 | 0.0270 | 0.1563 | 0.0290 | 0.0219 | 0.1157 | 0.0194 | 0.0157 |
| $S^{(1)}_{60:70}$ | 2.0279 | 0.5950 | 0.4131 | 2.0724 | 0.6015 | 0.4182 | 2.0429 | 0.5873 | 0.4051 | 2.0289 | 0.595 | 0.4118 |
| | 0.0406 | 0.0112 | 0.0075 | 0.0504 | 0.0116 | 0.0079 | 0.0425 | 0.0113 | 0.0075 | 0.0353 | 0.0090 | 0.0062 |
| $S^{(2)}_{60:70}$ | 2.0581 | 0.6116 | 0.4054 | 2.1114 | 0.6173 | 0.4092 | 2.0743 | 0.6036 | 0.3956 | 2.0553 | 0.6106 | 0.4049 |
| | 0.0772 | 0.0115 | 0.0095 | 0.0925 | 0.0122 | 0.0102 | 0.0795 | 0.0114 | 0.0095 | 0.0714 | 0.0095 | 0.0080 |
| $S^{(3)}_{60:70}$ | 2.0423 | 0.6183 | 0.4150 | 2.0911 | 0.6258 | 0.4204 | 2.0576 | 0.6105 | 0.4077 | 2.0411 | 0.6157 | 0.4127 |
| | 0.0809 | 0.0117 | 0.0106 | 0.0930 | 0.0124 | 0.0113 | 0.0830 | 0.0116 | 0.0107 | 0.0761 | 0.0097 | 0.0087 |

Table 3: The 95% ACILs and the corresponding CPs of parameters for different censoring schemes are reported, when $\alpha = 2$, $\beta_1 = 0.6$ and $\beta_2 = 0.4$.

| Scheme | MLE | | | Boot_p | | | Boot_T | | | Bayes MCMC | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\alpha$ | $\beta_1$ | $\beta_2$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\alpha$ | $\beta_1$ | $\beta_2$ | $\alpha$ | $\beta_1$ | $\beta_2$ |
| $S^{(1)}_{20:50}$ | 1.3794 | 0.8125 | 0.6178 | 1.4982 | 0.9263 | 0.6903 | 1.4252 | 0.9478 | 0.7671 | 1.3203 | 0.6855 | 0.5250 |
| | 0.964 | 0.942 | 0.961 | 0.961 | 0.923 | 0.944 | 0.925 | 0.963 | 0.947 | 0.962 | 0.961 | 0.956 |
| $S^{(2)}_{20:50}$ | 1.2843 | 0.7351 | 0.6158 | 1.4445 | 0.8925 | 0.7327 | 1.3598 | 0.8838 | 0.7601 | 1.2453 | 0.6326 | 0.5265 |
| | 0.933 | 0.925 | 0.941 | 0.962 | 0.962 | 0.953 | 0.936 | 0.906 | 0.943 | 0.951 | 0.954 | 0.935 |
| $S^{(3)}_{20:50}$ | 1.7227 | 0.9885 | 0.8007 | 1.9511 | 1.2812 | 0.9412 | 1.856 | 1.2036 | 0.9874 | 1.5017 | 0.6714 | 0.528 |
| | 0.906 | 0.887 | 0.943 | 0.833 | 0.868 | 0.887 | 0.962 | 0.849 | 0.925 | 0.943 | 0.962 | 0.934 |
| $S^{(1)}_{40:50}$ | 1.0828 | 0.5057 | 0.3937 | 1.2049 | 0.5490 | 0.4133 | 1.2669 | 0.5634 | 0.4236 | 1.1370 | 0.4698 | 0.3686 |
| | 0.981 | 0.925 | 0.906 | 0.943 | 0.943 | 0.925 | 0.981 | 0.887 | 0.906 | 0.962 | 0.925 | 0.943 |
| $S^{(2)}_{40:50}$ | 1.1888 | 0.5181 | 0.4064 | 1.2218 | 0.5534 | 0.4408 | 1.1445 | 0.5476 | 0.4242 | 1.0596 | 0.4788 | 0.3794 |
| | 0.943 | 0.925 | 0.943 | 0.962 | 0.925 | 0.961 | 0.943 | 0.906 | 0.911 | 0.943 | 0.962 | 0.932 |
| $S^{(3)}_{40:50}$ | 1.2201 | 0.5176 | 0.4150 | 1.2564 | 0.5669 | 0.4536 | 1.1644 | 0.5546 | 0.4655 | 1.0871 | 0.4783 | 0.3843 |
| | 0.925 | 0.943 | 0.943 | 0.943 | 0.951 | 0.937 | 0.925 | 0.934 | 0.947 | 0.925 | 0.962 | 0.943 |
| $S^{(1)}_{35:70}$ | 1.0582 | 0.5331 | 0.4358 | 1.2611 | 0.5831 | 0.4674 | 1.2353 | 0.6047 | 0.5039 | 1.1314 | 0.5007 | 0.4093 |
| | 0.962 | 0.943 | 0.943 | 0.925 | 0.981 | 0.925 | 0.943 | 0.943 | 0.906 | 0.943 | 0.981 | 0.962 |
| $S^{(2)}_{35:70}$ | 1.1164 | 0.5414 | 0.4377 | 1.1326 | 0.6020 | 0.4846 | 1.1088 | 0.5864 | 0.4870 | 1.0102 | 0.4942 | 0.4041 |
| | 0.981 | 0.925 | 0.943 | 0.925 | 0.943 | 0.962 | 0.981 | 0.943 | 0.887 | 0.943 | 0.962 | 0.981 |
| $S^{(3)}_{35:70}$ | 1.1205 | 0.5424 | 0.4493 | 1.3238 | 0.6311 | 0.4918 | 1.1933 | 0.5549 | 0.4612 | 1.1473 | 0.4967 | 0.4044 |
| | 0.962 | 0.925 | 0.981 | 0.906 | 0.906 | 0.962 | 0.943 | 0.925 | 0.906 | 0.925 | 0.943 | 0.956 |
| $S^{(1)}_{60:70}$ | 0.9407 | 0.4219 | 0.3324 | 1.0049 | 0.4428 | 0.3454 | 0.9986 | 0.4250 | 0.3462 | 0.9326 | 0.4014 | 0.3178 |
| | 0.962 | 0.981 | 0.981 | 0.925 | 0.943 | 0.983 | 0.962 | 0.925 | 0.962 | 0.962 | 0.943 | 0.981 |
| $S^{(2)}_{60:70}$ | 0.9294 | 0.4101 | 0.3304 | 1.0032 | 0.4326 | 0.3472 | 0.9648 | 0.4413 | 0.3431 | 0.9065 | 0.3904 | 0.3145 |
| | 0.981 | 0.967 | 0.962 | 0.943 | 0.966 | 0.962 | 0.981 | 0.981 | 0.967 | 0.981 | 0.973 | 0.962 |
| $S^{(3)}_{60:70}$ | 0.9163 | 0.4059 | 0.3274 | 0.9824 | 0.4308 | 0.3392 | 0.9559 | 0.4332 | 0.3524 | 0.8932 | 0.3891 | 0.3128 |
| | 0.925 | 0.962 | 0.954 | 0.962 | 0.950 | 0.968 | 0.943 | 0.943 | 0.956 | 0.943 | 0.939 | 0.958 |

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# 6 Data Analysis

In this section, we will provide real data, these data were presented by Hoel (1972) and Cramer and Schmiedt (2011), as well as analyzed by Modhesh and Abd-Elmougod (2015), and they proved that they follow the Burr XII model before going for more analyzes, the data was divided by 1000. It was obtained from a laboratory experiment in which male mice received a radiation dose of 300 roentgens. The cause of death for each mouse was determined by autopsy. Restricting the analysis to two causes of death, for the purpose of analysis, we consider reticulum cell sarcoma as cause 1 and combine the other causes of death as cause 2.

To check the validity of the model, we compute the the Kolmogorov Smirnov (K-S) statistic whether the Burr XII model is suitable for this data. The maximum likelihood estimates of $\alpha$ and $\beta$ based on the two causes of death are (8.1993, 35.6497) and (2.2261, 6.2144), respectively. In deaths due to cause 1 the K-S distance and the associated $p$-value are 0.0711 and 0.9907 , respectively, and for the deaths due to cause 2 the corresponding values are 0.1093 and 0.740. Based on the $p$-values, the Burr XII model is found to fit the data well.

Suppose that the pre-determined censoring scheme is given by $m = 30$ and $R_1 = R_2 = ... = R_5 = 4$, $R_6 = ... = R_{15} = 2$, $R_{16} = ... = R_{22} = 1$ $R_{23} = ... = R_{30} = 0$, then a progressive type II censored sample of size 30 out of 77 is obtained as (0.040,2), (0.042,2), (0.051,2), (0.062,2), (0.206,2), (0.222,2), (0.228,2), (0.252,2), (0.259,2), (0.282,2), (0.317,1), (0.318,1), (0.399,1), (0.407,2), (0.517,2), (0.549,1), (0.552,1), (0.564,2), (0.567,2), (0.594,1), (0.596,1), (0.619,2), (0.621,2), (0.628,1), (0.631,1), (0.636,1), (0.649,1), (0.686,2), (0.713,1), (0.763,2). There were $m_1 = 12$ deaths due to cause 1 and $m_2 = 18$ deaths due to cause 2. Progressive censoring in these kinds of experiments may be invaluable in obtaining information on growths of tumors in the mice. At the time of death of a particular mouse, other mice may be randomly selected and removed from the study. Autopsies on these mice may lead to information on the progression of the cancer over time.

In light of this result using the progressive type II censored sample of size described above, we plot the profile log-likelihood function (14) in Fig. 1. From the Fig. 1 it is clear that the profile log-likelihood function is unimodal and the MLE of $\alpha$ is close to 2.244. We start the iteration to solve the Equation (13) with $\alpha = 2.244$. We compute different estimates of the parameters $\alpha$, $\beta_1$ and $\beta_2$. ML estimates $(.)_{Ml}$, estimates using the bootstrap methods $(.)_{Boot\_p}$ and $(.)_{Boot\_t}$ based on 10000 bootstrap samples, and Bayes MCMC estimates $(.)_{B-MCMC}$ using 10000 MCMC samples and discard the first 1000 values as 'burn-in'. Also we compute the 95% confidence intervals and the corresponding lengths under diffirent methods of estimation. The results are given in Table 1.

*Curr. Sci. Int., 12(2): 102-116, 2023*
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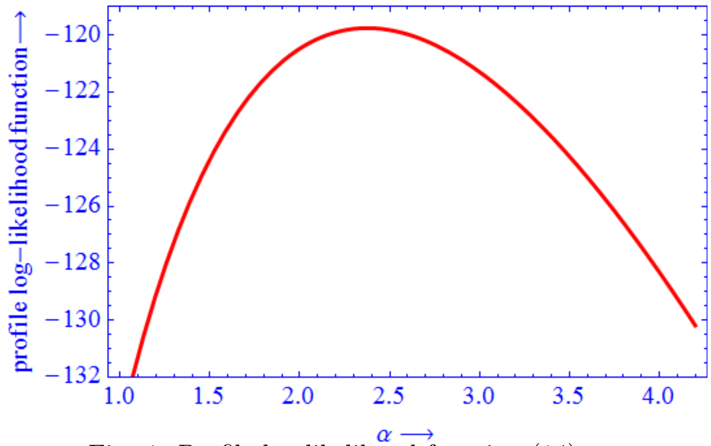


Fig. 1: Profile log-likelihood function (14).

Table 1: Point estimates, 95% confidence and credible intervals lengths for the Parameters.

| Methods of Estimation | Parameters | Estimates | Interval | Length |
|---|---|---|---|---|
| $(.)_{ML}$ | $\alpha$ | 2.3756 | (1.7279, 3.0234) | 1.2955 |
| | $\beta_1$ | 1.4571 | (0.4651, 2.4490) | 1.9840 |
| | $\beta_2$ | 2.1856 | (0.8801, 3.4911) | 2.6110 |
| $Boot\_P$ | $\alpha$ | 2.4520 | (1.9462, 3.1275) | 1.1813 |
| | $\beta_1$ | 1.5903 | (0.7433, 2.8924) | 2.1492 |
| | $\beta_2$ | 2.4585 | (1.3467, 4.3386) | 2.9920 |
| $Boot\_t$ | $\alpha$ | 2.4263 | (1.7374, 3.0901) | 1.3527 |
| | $\beta_1$ | 1.4353 | (0.1510, 2.2304) | 2.0794 |
| | $\beta_2$ | 2.2397 | (0.7396, 3.3101) | 2.5704 |
| $(.)_{Bayes\_MCMC}$ | $\alpha$ | 2.3749 | (1.7569, 3.0853) | 1.3284 |
| | $\beta_1$ | 1.4811 | (0.6628, 2.7838) | 2.1210 |
| | $\beta_2$ | 2.2210 | (1.1281, 3.9216) | 2.7935 |

# 7 Conclusions

In this paper, we have analyzed progressive type-II censored competing risks data. In particular, we have assumed that the latent failure times under the competing risks follow independent Burr XII distributions with common the shape parameters. We compared different statistical inference procedures and the performance of the unknown parameters based on MLE, Boot-p, Boot-t and Bayes methods in this setting. A numerical example has been presented to illustrate all the methods of inference developed in this paper. We have then conducted a simulation study to assess the performance of all these procedures.

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