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# Pre Topological Approximations of Rough Sets

[1]**Rmadan A. A.,** [2]**M. A. Elgyar and** [1]**M. Salama**

[1]*Mathematics Department, Faculty of Science, Beni Suef, Egypt*
[2]*Mathematics Department, Faculty of Science, Helwan University, Egypt*

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## ABSTRACT

The present contribution initiates two new operators based on the concepts of pre closure and pre interior. The range of membership of an object to a rough set has been enlarged to four choices instead of two in original rough set model. Many properties of the suggested operators are obtained. Examples and counter examples are given, in addition to examples for the comparison between the introduced concepts and their corresponding ones are also constructed.

***Key words:*** Pre closure, Pre interior, Pre lower, Pre upper, Rough Sets

## Introduction

In the last three decades of the twentieth century many new types of closure and interior operators have been introduced to topological spaces, such as pre closure and pre interior operators. But we can not use these types in Pawlak's space since every open set is closed in this space whose topology is based on equivalence classes .

Our approach to this problem is the generalization the equivalence relation in Pawlak's model by a general relation and generating a topological structure in the same way as in (Lashin *et al.,* 2005). We use the notions of pre closure and pre interior operators in their simplest form to construct new approximations called pre lower and pre upper approximations.

In (Lashin *et al.,* 2005), a general topological structure was used to define two new lower and upper approximations based on the notions of interior and closure operators (general upper and general lower approximations). However, in our approach we use new deferent approximations depending on pre closure and pre interior operators. Also, we obtain some properties of previous types of approximations (general lower, general upper). In addition, in (Lashin *et al.,* 2005) there are two membership relations, but in our approach the range of membership relations was increased to four.

## Fundamentals of the Pawlak's rough sets

Let $U$ be a finite set of objects called the Universe, $E \subseteq U \times U$ be an equivalence relation on $U$ , the pair $K = (U, E)$ is called approximation space, equivalence classes of the relation R are called elementary sets in $U$ . Let $X \subseteq U$ be a subset of $U$ , lower approximation and upper approximation of X in U denoted $\underline{E}$ and $\overline{E}$ are defined as follows:

$\underline{E}(X) = \{x \in U : \|[x]_E \subseteq X\}$ and $\overline{E}(X) = \{x \in U : \|[X]_E \subseteq \cap X \neq \varnothing\}$

The objects in $\underline{E}(X)$ can be classified as members surely belong to $X$ on the basis of knowledge in $R$ , while the objects in $\overline{E}(X)$ can be only classified as possible members of X using knowledge in E. We denote the boundary of X in E as, $BND_E(X) = \overline{E}(X) - \underline{E}(X)$ . If

**Corresponding Author:** M. Salama, Mathematics Department, Faculty of Science, Beni Suef, Egypt
E-mail: ms84math@yahoo.com

$BND_E(X) = \varnothing$ , we say that $X$ is definable or exact in $U$ , otherwise $X$ is said to be a non-definable set or rough. The family of all rough sets is denoted by $\zeta$ .

Let $\varnothing$ be the empty set, $X^c$ the complement of $X$ in $U$ , we have the properties of the Pawlak's rough sets in (Pawlak, 1991)

**Generalization of approximations**

The purpose of this article is to use general topological structures for introducing new approximation operators to theory of rough sets for topologies initiated in Tamer (Lashin *et al.,* 2005). It should be noted that Lin used Frechet topologies (Sierpenski and Krieger, 1952) and general binary relations (Lin, 1992) for the approximation concepts.

**Definition (3-1):-**

A general knowledge base is system $K = (U, R)$ where $U \neq \varnothing$ is a universal set and R be a family of general relations over U. Each relation R of R will be called a similarity relation and will be denoted by $SIM(K)$ and for each $x \in U, R(x) = \{y : xRy\}$ .
The topology $\tau$ generated by the sub base $S = \{R(x) : x \in U\}$ is not generally Pawlak topology; it coincides with it if $R$ is an equivalence.

**Definition (3-2):-**

Let $K = (U, R)$ be a general knowledge base and $R \in PRE(K)$. Let $X, Y \subseteq U$ , the general lower and upper approximations will be defined as follows $L(X) = \underline{R}(X) = X^0 = \cup\{G : G \in \tau$ and $G \subseteq X\}, U(X) = \overline{R}(X) = \overline{X} = \cap\{F : F \in \tau^c$ and $F \supseteq X\}$
The following proposition insures some properties of general lower and upper approximations

**Proposition 3.1:-**

Let $K = (U, R)$ be a general knowledge base and $R \in PRE(K)$. Let $X, Y \subseteq U$ then

(1) $\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$

(2) $\underline{R}\varnothing = \overline{R}\varnothing = \varnothing, \underline{R}U = \overline{R}U = U$

(3) $\overline{R}(X \cup Y) = \overline{R}X \cup \overline{R}U$

(4) $\underline{R}(X \cap Y) = \underline{R}X \cap \underline{R}U$

(5) $X \subseteq Y$ implies $\underline{R}X \subseteq \underline{R}Y$

(6) $X \subseteq Y$ implies $\overline{R}X \subseteq \overline{R}Y$

(7) $\underline{R}(X \cup Y) \supseteq \underline{R}X \cup \underline{R}U$

(8) $\overline{R}(X \cap Y) \subseteq \overline{R}X \cap \overline{R}U$

(9) $\underline{R}(X^c) = (\overline{R}X)^c$

(10) $\overline{R}(X^c) = (\underline{R}X)^c$

(11) $\underline{R}(\underline{R}(X)) = \overline{R}(\underline{R}(X)) = \underline{R}(X)$

(12) $\overline{R}(\overline{R}(X)) = \underline{R}(\overline{R}(X)) = \overline{R}(X)$

***Remark*** In General $\overline{R}(\underline{R}(X)) \neq \overline{R}(X)$ and $\underline{R}(\overline{R}(X)) \neq \overline{R}(X)$. The following example insures this fact.

***Example (3-1):-***

Consider $U=\{x_1, x_2, x_3, x_4, x_5, x_6\}$, the decision information system presented in Table which contains the set of condition attributes $C = \{a, b, c, f\}$, and the set of objects $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and construct the similarity matrix $M_a = \{a_{ij} = |a(x_i) - a(x_j)|\}$

| Attributes / Objects | $a$ | $b$ | $c$ | $f$ | $a$ | $b$ |
|---|---|---|---|---|---|---|
| $x_1$ | 81 | 77 | 84 | 83 | $x_1$ | 81 |
| $x_2$ | 100 | 81 | 93 | 85 | $x_2$ | 100 |
| $x_3$ | 71 | 78 | 89 | 60 | $x_3$ | 71 |
| $x_4$ | 93 | 82 | 88 | 60 | $x_4$ | 93 |
| $x_5$ | 97 | 87 | 91 | 85 | $x_5$ | 97 |
| $x_6$ | 93 | 68 | 87 | 90 | $x_6$ | 93 |

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| $a$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| $x_1$ | 0 | 19 | 10 | 12 | 16 | 12 |
| $x_2$ | 19 | 0 | 29 | 7 | 3 | 7 |
| $x_3$ | 10 | 29 | 0 | 22 | 26 | 22 |
| $x_4$ | 12 | 7 | 22 | 0 | 4 | 0 |
| $x_5$ | 16 | 3 | 26 | 4 | 0 | 4 |
| $x_6$ | 12 | 7 | 22 | 0 | 4 | 0 |

$x_i R_a x_j \Leftrightarrow |a(x_i) - a(x_j)| < 5$.

$R(x_1) = \{x_1\}, R(x_2) = \{x_2, x_5\}, R(x_3) = \{x_3\}, R(x_4) = \{x_4, x_5, x_6\},$
$R(x_5) = \{x_2, x_4, x_5, x_6\}, R(x_6) = \{x_4, x_5, x_6\}$

$\tau = \{U, \emptyset, \{x_1\}, \{x_2, x_5\}, \{x_3\}, \{x_4, x_5, x_6\}, \{x_2, x_4, x_5, x_6\}, \{x_1, x_2, x_5\}, \{x_1, x_3\}, \{x_1, x_4, x_5, x_6\},$
$\{x_1, x_2, x_4, x_5, x_6\}, \{x_1, x_2, x_3, x_5\}, \{x_1, x_3, x_4, x_5, x_6\}, \{x_2, x_3, x_5\}, \{x_3, x_4, x_5, x_6\}$
$, \{x_2, x_3, x_4, x_5, x_6\}, \{x_5\}, \{x_1, x_5\}, \{x_3, x_5\}, \{x_1, x_3, x_5\}\}$
$\tau^c = \{\emptyset, U, \{x_2, x_3, x_4, x_5, x_6\}, \{x_1, x_3, x_4, x_6\}, \{x_1, x_2, x_4, x_5, x_6\}, \{x_1, x_2, x_3\}$
$, \{x_1, x_3\}, \{x_3, x_4, x_6\}, \{x_2, x_4, x_5, x_6\}, \{x_2, x_3\}, \{x_3\}, \{x_4, x_6\}, \{x_2\}, \{x_1, x_4, x_6\}$
$, \{x_1, x_2\}, \{x_1\}\}$
Let $X = \{x_2, x_3\}$

so $\overline{R}(X) = \{x_2, x_3\}, \underline{R}(X) = \{x_3\}$, $\overline{R}(\underline{R}(X)) = \{x_3\} \neq \{x_2, x_3\} = \overline{R}(X)$ .

Also $\underline{R}(\overline{R}(X)) = \{x_3\} \neq \{x_2, x_3\} = \overline{R}(X)$ .

**Pre approximation concepts**

Closure and interior operators played a significant role in the approximation process. In the last three decades of the twentieth century, new types of closure and interior may have been introduced to the topological space. But as far as we know, they were not used in the approximations. In this section, we initiate the use of pre closure and pre interior in this process.

***Pre rough sets***

Since $\overline{R}(\underline{R}(X)) \neq \overline{R}(X)$, and $\underline{R}(\overline{R}(X)) \neq \overline{R}(X)$, we introduce the following definition .

***Definition (4-1-1):***

Let $(U, R)$ be a general knowledge base and $R \in pre(K)$ . let $X \subseteq U$ , we define the following sets

$\underline{R}_p(X) = P.L(X) = X \cap \underline{R}(\overline{R}(X)),$

$\overline{R}^p(X) = P.U(X) = X \cup \overline{R}(\underline{R}(X)).$

where PL(X) and PU(X) are called R-pre lower and R-pre upper of X .

***Example (4-1-1):-***

from Example (3-1)

| $X$ | $\overline{R}(\underline{R}(X))$ | $\underline{R}(\overline{R}(X))$ | $\underline{R}_P(X)$ | $\overline{R}^P(X)$ |
|---|---|---|---|---|
| $\{x_1\}$ | $\{x_1\}$ | $\{x_1\}$ | $\{x_1\}$ | $\{x_1\}$ |
| $\{x_3\}$ | $\{x_3\}$ | $\{x_3\}$ | $\{x_3\}$ | $\{x_3\}$ |
| $\{x_1,x_3\}$ | $\{x_1,x_3\}$ | $\{x_1,x_3\}$ | $\{x_1,x_3\}$ | $\{x_1,x_3\}$ |
| $\{x_2,x_4,x_5,x_6\}$ | $\{x_2,x_4,x_5,x_6\}$ | $\{x_2,x_4,x_5,x_6\}$ | $\{x_2,x_4,x_5,x_6\}$ | $\{x_2,x_4,x_5,x_6\}$ |
| $\{x_1,x_2,x_4,x_5,x_6\}$ | $\{x_1,x_2,x_4,x_5,x_6\}$ | $\{x_1,x_2,x_4,x_5,x_6\}$ | $\{x_1,x_2,x_4,x_5,x_6\}$ | $\{x_1,x_2,x_4,x_5,x_6\}$ |
| $\{x_2,x_3,x_4,x_5,x_6\}$ | $\{x_2,x_3,x_4,x_5,x_6\}$ | $\{x_2,x_3,x_4,x_5,x_6\}$ | $\{x_2,x_3,x_4,x_5,x_6\}$ | $\{x_2,x_3,x_4,x_5,x_6\}$ |

In $U$ , $X$ is said to be pre exact if $\overline{R}^p(X) = \underline{R}_p(X)$ otherwise $X$ is said to be pre rough.

***Example (4-1-2):-***

from Example (3-1) Let $X = \{x_1, x_3\}$

so $\overline{R}(X)=\{x_1,x_3\}, \underline{R}(X) = \{x_1,x_3\}$, $\overline{R}(\underline{R}(X)) = \{x_1,x_3\}$ .

and $\underline{R}(\overline{R}(X)) = \{x_1,x_3\} \Rightarrow \underline{R}_p(X) = \{x_1,x_3\} = \overline{R}^p(X) = \{x_1,x_3\}$

so $X$ is pre exact

***Example (4-1-3):-***

from Example (2-1) let $X = \{x_1, x_3, x_4\}$ ,so $\underline{R}(X) = \{x_1,x_3\}$ ,

$\overline{R}(X) = \{x_1,x_3,x_4,x_6\}$ ,$\overline{R}(\underline{R}(X)) = \{x_1,x_3\}$ , $\underline{R}(\overline{R}(X)) = \{x_1,x_3\}$

so $\underline{R}_p(X) = \{x_1,x_3\}$ , on the other hand $\overline{R}^p(X) = \{x_1,x_3,x_4\}$

The following proposition insures the relation between exact and pre exact set.

***Proposition (4-1-1):***

Let $K = (U, R)$ be a general knowledge base,

$X \subseteq U$ and $R \in pre(K)$, So if is R-exact, then X is pre R-exact .

***Proof***

If $\underline{R}X = \overline{R}X$ then

$$\underline{R}_p(X) = X \cap \underline{R}(\overline{R}(X)) = X \cap \underline{R}(\underline{R}(X)) = X \cap \underline{R}(x) = X$$

and $\overline{R}^{p}(X) = X \cup \overline{R}(\underline{R}(X)) = X \cup \overline{R}(\overline{R}(X)) = X \cup \overline{R}(X) = X$

$\Rightarrow \overline{R}^{p}(X) = \underline{R}_{p}(X)$

So X is pre R-exact

The following example shows that the converse of the previous proposition is not generally true,

***Example (4-1-4):-***

Let $U = \{x_1, x_2, x_3, x_4\}$ and R be a general relation on U has the following

$U \setminus R = \{\{x_1, x_2\}, \{x_2, x_3, x_4\}, \{x_3\}\}$ , $\tau = \{U, \varnothing, \{x_1, x_2\}, \{x_2, x_3, x_4\},$

$\{x_3\}, \{x_2\}, \{x_1, x_2, x_3\}, \{x_2, x_3\}\}$ , $\tau^c = \{\varnothing, U, \{x_3, x_4\}, \{x_1\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}$

$, \{x_4\}, \{x_1, x_4\}\}$ ,let $X = \{x_3, x_4\}$ , $\underline{R}(X) = \{x_3\}$ , $\overline{R}(X) = \{x_3, x_4\}$ ,and

$\overline{R}(\underline{R}(X)) = \{x_3, x_4\}$ , $\underline{R}(\overline{R}(X)) = \{x_3\}$

so $\underline{R}_p(X) = \{x_3, x_4\}$ , also $\overline{R}^{p}(X) = \{x_3, x_4\}$ , i.e $\underline{R}_p(X) = \overline{R}^{p}(X)$ but $\underline{R}(X) \neq \overline{R}(X)$

**Remark** The previous proposition indicates that the class of pre exact contains the class of exact sets and this increases the accuracy of knowledge.

The following proposition illustrate he properties of pre lower and pre upper approximations.

***Proposition (4-1-2)*** :

Let $K = (U, R)$ be a general knowledge base and $R \in pre(K)$. let $X \subseteq U$ then :

1) $\underline{R}(X) \subseteq \underline{R}_p(X) \subseteq X \subseteq \overline{R}^{p}(X) \subseteq \overline{R}(X)$

2) $\underline{R}(\underline{R}_p(X)) = \underline{R}_p(\underline{R}(X)) = \underline{R}(X)$

3) $\overline{R}(\overline{R}^{p}(X)) = \overline{R}^{p}(\overline{R}(X)) = \overline{R}(X)$

***Proof***

1) $X^{o} \subseteq X$ and $X^{o} \subseteq X^{-o} \Rightarrow X^{o} \subseteq X \cap X^{-o}$

$\Rightarrow \underline{R}(X) \subseteq \underline{R}_p(X)$ , also since $X \subseteq (X \cap X^{o-})$

and $X^{o-} \subseteq X^{-}$ then $X \subseteq \overline{R}^{p}(X) \subseteq \overline{R}(X)$

2) $\underline{R}(\underline{R}_p(X)) = \underline{R}(X \cap X^{-o}) = (X \cap X^{-o})^{o} = X^{o} \cap X^{-oo}$

since $X^{o} \subseteq X^{-o} \Rightarrow X^{o} = X^{oo} \subseteq X^{-oo}$

$\Rightarrow (X^{o} \cap X^{-oo}) = X^{o} = \underline{R}(X) \Rightarrow \underline{R}(\underline{R}_p(X)) = X^{o} = \underline{R}(X)$ on the other hand ,

$\underline{R}_p(\underline{R}X) = X^{o} \cap X^{o-o} = \underline{R}X \Rightarrow \underline{R}(\underline{R}_p X) = \underline{R}_p(\underline{R}X) = \underline{R}X$

3) By a similar proof.

***Proposition (4-1-3) :*** Let $K = (U, R)$ be a general knowledge base and $R \in pre(K)$ .

let $X, Y \subseteq U$ then :

1) $\underline{R}_p \varnothing = \varnothing, \underline{R}_p U = U$

2) $\underline{R}_p(X \cap Y) \subseteq \underline{R}_p(X) \cap \underline{R}_p(Y)$

3) If $X \subseteq Y \Rightarrow \underline{R}_p(X) \subseteq \underline{R}_p(Y)$

4) $\underline{R}_p(X \cup Y) \supseteq \underline{R}_p(X) \cup \underline{R}_p(Y)$

***Proof***

1) $\underline{R}_p \varnothing = \varnothing \cap \varnothing^{-o} = \varnothing, \underline{R}_p U = U$ ( obvious )

2) $\underline{R}_p(X \cap Y) = (X \cap Y) \cap (X \cap Y)^{-o} \subseteq (X \cap Y) \cap (X^{-} \cap Y^{-})^{o} \subseteq (X \cap Y) \cap$

$(X^{-o} \cap Y^{-o}) = (X \cap X^{-o}) \cap (Y \cap Y^{-o})$

$\Rightarrow \underline{R}_p(X \cap Y) \subseteq \underline{R}_p(X) \cap \underline{R}_p(Y)$

3) $X \subseteq Y \Rightarrow X^{-} \subseteq Y^{-} \Rightarrow X^{-o} \subseteq Y^{-o} \Rightarrow X \cap X^{-o} \subseteq Y \cap Y^{-o}$

$\Rightarrow \underline{R}_p(X) \subseteq \underline{R}_p(Y)$

4) $\underline{R}_p(X \cup Y) = (X \cup Y) \cap (X \cup Y)^{-o} \supseteq (X \cup Y) \cap (X^{-o} \cup Y^{-o})$

$= ((X \cup Y) \cap X^{-o}) \cup ((X \cup Y) \cap Y^{-o})$

$= ((X \cap X^{-o}) \cup (Y \cap X^{-o})) \cup ((X \cap Y^{-o}) \cup (Y \cap Y^{-o}))$

$\supseteq (X \cap X^{-o}) \cup (Y \cap Y^{-o}) = \underline{R}_p(X) \cup \underline{R}_p(Y)$

$\Rightarrow \underline{R}_p(X \cup Y) \supseteq \underline{R}_p(X) \cup \underline{R}_p(Y)$

***Proposition (4-1-4) :*** Let $K = (U, R)$ be a general knowledge base and $R \in pre(K)$ . let $X, Y \subseteq U$ then :

1) $\overline{R}^p \varnothing = \varnothing, \overline{R}^p U = U$

2) $\overline{R}^p(X \cup Y) \supseteq \overline{R}^p(X) \cup \overline{R}^p(Y)$

3) $X \subseteq Y \Rightarrow \overline{R}^p(X) \subseteq \overline{R}^p(Y)$

4) $\overline{R}^p(X \cap Y) \subseteq \overline{R}^p(X) \cap \overline{R}^p(Y)$

***Proof***

1) $\overline{R}^p \varnothing = \varnothing \cup \varnothing^{o-} = \varnothing, \overline{R}^p U = U \cup U^{o-} = U$

2) $\overline{R}^p(X \cup Y) = (X \cup Y) \cup (X \cup Y)^{o-} \supseteq (X \cup Y) \cup (X^{o-} \cup Y^{o-}) = (X \cup X^{o-}) \cup (Y \cup Y^{o-})$

$\Rightarrow \overline{R}^p(X \cup Y) \supseteq \overline{R}^p(X) \cup \overline{R}^p(Y)$

3) $X \subseteq Y \Rightarrow X^o \subseteq Y^o \Rightarrow X^{o-} \subseteq Y^{o-} \Rightarrow X \cup X^{o-} \subseteq Y \cup Y^{o-}$

$\Rightarrow \overline{R}^p(X) \subseteq \overline{R}^p(Y)$

4) $\overline{R}^p(X \cap Y) = (X \cap Y) \cup (X \cap Y)^{o-} = (X \cap Y) \cup (X^o \cap Y^o)^-$

$\subseteq (X \cap Y) \cup (X^{o-} \cap Y^{o-}) = ((X \cap Y) \cup X^{o-}) \cap ((X \cap Y) \cup Y^{o-})$

$\subseteq (X \cup X^{o-}) \cap (Y \cup Y^{o-}) = \overline{R}^p(X) \cap \overline{R}^p(Y)$

***Proposition (4-1-5) :*** Let $K = (U, R)$ be a general knowledge base and $R \in pre(K)$ . let $X, Y \subseteq U$ then :

1) $\underline{R}_p (X^c) = (\overline{R}^p (X))^c$

2) $\overline{R}^p (X^c) = (\underline{R}_p (X))^c$

***Proof***

1) $\underline{R}_p (X^c) = X^c \cap (X^c)^{-o} = X^c \cap ((X^o)^c) = X^c \cap (X^{o-})^c = (X \cup X^{o-})^c$

$= (\overline{R}^p X)^c$

2) $\overline{R}^p (X^c) = X^c \cup (X^c)^{o-} = X^c \cup ((X^-)^c)^- = X^c \cup (X^{-o})^c = (X \cap (X^{-o}))^c$

$= (\underline{R}_p X)^c$

## Conclusion

The suggested approximations help in the process of knowledge discovery and decision making in information systems in general and specially for universes in which equivalence relations give discrete structures. The similarity relation constructed helps in solving such limitation.

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