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Anti-Interval Valued Fuzzy H-Ideals of BCI-Algebras

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ABSTRACT

In this paper, we introduce the notion of anti- interval valued fuzzy H- ideal of BCI-algebras, and study some of their properties. We show that an interval valued fuzzy subset of a BCI-algebra is a fuzzy H- ideal if and only if the complement of this interval valued fuzzy subset is an anti- interval valued fuzzy H-ideal, and the Cartesian product of tow anti-interval valued fuzzy H-ideals is anti-interval valued fuzzy H- ideals We investigate how to deal with the homomorphic image (pre-image) of anti- interval valued H- ideal of BCI-algebra. Moreover, we introduce the notion of Cartesian product of anti- interval valued fuzzy H- ideals and then we study some related properties

Key words: Ideal of BCI- algebras , anti- interval valued fuzzy H- ideal , homomorphic image (pre-image) of H- ideal , Cartesian product of anti- interval valued fuzzy H- ideal

Introduction

The concept of a fuzzy set was introduced by Zadeh, (1965) and was used afterwards by many other outhers in various branches of mathematics. In 1966, Ise′ki and Tanaka,(1978) introduced the notion of BCI-algebras. Xi, (1991) applied the concept of fuzzy set to BCI-algebras and gave some properties of it. After that Jun and Meng investigated further properties of fuzzy BCI-algebras and fuzzy ideal [see(Meng and X. L. Xin, 1992; Biswas,1990; Jun,1993) . Biswas, (1990) introduced the concept of anti- interval valued fuzzy sub-group. Modifying this idea, in this paper, we introduce the concept of anti- interval valued fuzzy H- ideal of BCI-algebra and investigate some related properties. We show that an interval valued fuzzy subset of a BCI-algebra is a fuzzy H- ideal if and only if the complement of this interval valued fuzzy subset is an anti- interval valued fuzzy H-ideal, and the Cartesian product of tow anti-interval valued fuzzy H-ideals is anti-interval valued fuzzy H- ideals We investigate how to deal with the homomorphic image (pre-image) of anti- interval valued H- ideal of BCI-algebra. Moreover, we introduce the notion of Cartesian product of anti- interval valued H- ideals and then we study some related properties, we characterize anti- interval valued fuzzy H- ideal by it.

Preliminaries

Definition (Huang , 1999)

An algebra (X,*,0) of type (2,0) is called a BCI-algebra if it Satisfies the following axioms:

(1) $((x*y)*(x*z))*(z*y)=0$
(2)$(x*(x*y))*y=0$
(3)$x*x=0$
(4)$x*y=0$ *and* $y*x=0$ *imply* $x=y$

In BCI-algebra, we can define a partial ordering"$\leq$ " by $x\leq y$ if and only if $x*y=0$

Proposition (Xueling *et al.,* 2008)

A BCI-algebra X satisfies the following properties:

(1.1)$(x*y)*z=(x*z)*y$
(1.2)$x*0=x$
(1.3)$0*(x*y)=(0*x)*(0*y)$
(1.4)$0*(0*(x*y))=0*(y*x)$
(1.5)$(x*z)*(y*z)\leq x*y$
(1.6)$x*y=0$ *implies* $x*z\leq y*z$ *and* $z*y\leq z*x$
(1.7)$(x*y)*(x*z)\leq z*y$
(1.8)$x*(x*(x*y))=x*y$

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Definition (Xi, 1999).

A non-empty subset A of a BCI-algebra (X,*, 0) is called an ideal of X if for any $x, y \in X$ the following conditions hold

(i)o ∈ A

$(ii) x * y \in A$ *and* $y \in A$ *imply that* $x \in A$

Definition **(**Khalid and Ahmad,1999)

A non-empty subset A of BCI-algebra (X.*,0) is called an H-ideal of X ,if for any $x, y, z \epsilon X$,the following conditions hold

(a)o∈A

$(b)(x * (y * z)) \in A$ *and* $y \epsilon A$ *imply that* $x * z \in A$

If we put z=0, then it follows that A is an ideal, thus every H-ideal is an ideal

Definition**:** (Meng and Xin,1992)

for all $x \in X$ .then the interval valued fuzzy set is given by $A = \{(x, \tilde{\mu}_A(x)), x \in X\}$,where $\tilde{\mu}\ (x) : X \to D[0,1]$,

D [0, 1] denote the family of all closed subintervals of [0, 1]

Now, we begin with the concepts of interval-valued fuzzy sets.

An interval number is $\tilde{a} = [a_L, a_U]$, where $0 \le a_L \le a_U \le 1$ .Let D [0, 1] denote the family of all closed subintervals of [0, 1], i.e.,

$$D[0,1] = \{\tilde{a} = [a_L, a_U] : a_L \le a_U \quad for\ a_L, a_U \in I\}.$$

We define the operations $\le$, $\ge$, $=$, r min and r max in case of two elements in D[0, 1]. We consider two elements $\tilde{a} = [a_L, a_U]$ and $\tilde{b} = [b_L, b_U]$ in D [0, 1].

Then:

1- $\tilde{a} \le \tilde{b} \quad iff \ a_L \le b_L, a_U \le b_U$;

2- $\tilde{a} \ge \tilde{b} \quad iff \ a_L \ge b_L, a_U \ge b_U$;

3- $\tilde{a} = \tilde{b} \quad iff \ a_L = b_L, a_U = b_U$;

4- $\tilde{a} \wedge \tilde{b} = rmim\{\tilde{a}, \tilde{b}\} = [\min\{a_L, b_L\}, \min\{a_U, b_U\}]$;

5- $\tilde{a} \vee \tilde{b} = r\max\{\tilde{a}, \tilde{b}\} = [\max\{a_L, b_L\}, \max\{a_U, b_U\}]$

Here we consider that $\tilde{0} = [0,0]$ as least element and $\tilde{1} = [1,1]$ as greatest element.

Let $\tilde{a}_i \in D[0,1]$,where $i \in \Lambda$ .We define

$$r\inf_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda}(a_i)_L, \inf_{i\in\Lambda}(a_i)_U\right] \text{ and } r\sup_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda}(a_i)_L, \sup_{i\in\Lambda}(a_i)_U\right]$$

An interval valued fuzzy set (briefly, i-v-f-set) $\widetilde{\mu}$ on X is defined as

$\widetilde{\mu} = \{\langle x\ ,\ [\mu_L(x), \mu_U(x)],\ x \in X \rangle\}$, where $\widetilde{\mu} : X \to D[0,1]$ and $\mu_L(x) \le \mu_U(x)$,for all

$x \in X$. Then the ordinary fuzzy sets $\mu^N(x * y) \le \mu^N(x)$ and $\mu_U : X \to [0,1]$ are called a elower fuzzy

set and an upper fuzzy set of $\widetilde{\mu}$ respectively

Definition.

An interval valued fuzzy set $\tilde{\mu}$ in X has the sup property if for any subset T⊆X ,there exists x_0 ∈T such that $\tilde{\mu}(x_0) = rsup\tilde{\mu}(t)$, t∈T

Definition .

An interval valued fuzzy set $\tilde{\mu}$ in X has the inf property if for any subset T⊆X ,there esxists x_0 ∈T such that $\tilde{\mu}(x_0) = rinf\ \tilde{\mu}(t)$, t∈T

Definition .

An interval valued fuzzy set $\tilde{\mu}$ of a BCI-algebra X is called an anti- interval valued fuzzy sub-algebra of X if :

$\tilde{\mu}\ (x\ *\ y)\ \le r\ max\{\tilde{\mu}\ (x)\ ,\tilde{\mu}\ (y)\}\ for\ all\ x\ ,y\ \in\ X.$

Definition .

An interval valued fuzzy set $\tilde{\mu}$ of a BCI-algebra X is called an anti- interval valued fuzzy ideal of X if it satisfies:

$(F1)\ \tilde{\mu}\ (0)\ \le\ \tilde{\mu}\ (x),$

$(F2)\ \tilde{\mu}\ (x)\ \le r\ max\ \{\tilde{\mu}\ (x\ *\ y), \tilde{\mu}\ (y)\}, for\ all\ x\ ,y \in\ X.$

Proposition.

Every anti- interval valued fuzzy ideal $\tilde{\mu}$ of a BCI-algebra X is order preserving

***Definition.* (**Khalid and Ahmad, 1999**).**

A fuzzy subset μ of a BCI-algebra X is called a fuzzy H-ideal if it satisfies:

$(a)\ \mu(0) \ge \mu(x)$

$(b) \mu(x * z) \ge min\{\mu(x * (y * z)), \mu(y)\}\ for\ all\ x, y, z \in X$

Proposition.

Let $\tilde{\mu}$ be an anti- interval valued fuzzy ideal of a BCI-algebra X .then $x * y \le z$ implies $\tilde{\mu}(x) \le rmax\{\tilde{\mu}(y), \tilde{\mu}(z)\}\ for\ all\ x, y, z \in X$

Anti- interval valued Fuzzy H-ideals of BCI-algebra

Definition.

An interval valued fuzzy subset $\tilde{\mu}$ of a BCI-algebra X is called an anti- interval valued fuzzy H-ideal if it satisfies:

$(F_1) \tilde{\mu}(0) \le \tilde{\mu}(x)$

$(F_3) \tilde{\mu}(x * z) \le rmax\{\tilde{\mu}(x * (y * z)), \tilde{\mu}(y)\}\ for\ all\ x, y, z \in X$ Clearly if z=0 gives $\tilde{\mu}$ is an anti- interval valued fuzzy ideal of X

Example .If X= {0, l, m, n,p,q} define * in X is

*	0	l	m	n	p	q
0	0	0	0	n	n	n
l	l	0	l	p	n	p
m	m	m	0	q	q	n
n	n	n	n	0	0	0
p	p	n	p	l	0	l
q	q	q	n	m	m	0

Define $\tilde{\mu}_{\tilde{A}}$: X⟶ D[0,1] as $\tilde{\mu}_{\tilde{A}}(0) = \tilde{t}_1$,$\tilde{\mu}_{\tilde{A}}$(l)=$\tilde{t}_2$,$\tilde{\mu}_{\tilde{A}}$(m)=$\tilde{\mu}_{\tilde{A}}$(q)=$\tilde{\mu}_{\tilde{A}}$(p)=$\tilde{\mu}_{\tilde{A}}$(n)=$\tilde{t}_3$

For $\tilde{t}_1 < \tilde{t}_2 < \tilde{t}_3$, where $\tilde{t}_i \epsilon D[0,1], i = 1,2,3$ then routine calculations show that $\tilde{A}$ is an anti- interval valued fuzzy H-ideal of X

Proposition.

If $\tilde{A}$ is a anti- interval valued fuzzy H-ideal with membership function $\tilde{\mu}_{\tilde{A}}$ in a BCI-algebra X, then $\tilde{\mu}_{\tilde{A}}(x * z) \leq \tilde{\mu}_{\tilde{A}}(x * (0 * z))$.in particular, $\tilde{\mu}_{\tilde{A}}(0 * z) \leq \tilde{\mu}_{\tilde{A}}(0 * (0 * z))$ $for\ all\ z\ in\ X$

Theorem

In an associative BCI-algebra X, every anti- interval valued fuzzy ideal is an anti- interval valued fuzzy H-ideal of X

Proof.

Since $\tilde{\mu}_{\tilde{A}}(x * (y * z)) \vee \tilde{\mu}_{\tilde{A}}(y) = \tilde{\mu}_{\tilde{A}}((x * y) * z) \vee \tilde{\mu}_{\tilde{A}}(y) = \tilde{\mu}_{\tilde{A}}((x * z) * y) \vee \tilde{\mu}_{\tilde{A}}(y) \geq \tilde{\mu}_{\tilde{A}}(x * z)$ (A is a anti- interval valued fuzzy ideal) or $\tilde{\mu}_{\tilde{A}}(x * z) \leq \tilde{\mu}_{\tilde{A}}(x * (y * z)) \vee \tilde{\mu}_{\tilde{A}}(y)$, wich shows that $\tilde{A}$ is an anti- interval valued fuzzy H-ideal of X this completes the proof

Remark.

It is easily seen that the intersection of any number of anti- interval valued fuzzy H-ideal is an anti- interval valued fuzzy H-ideal of X

Theorem.

Let $\tilde{\mu}$ be an interval valued fuzzy set of a BCI-algebra X then $\tilde{\mu}$ is an anti- interval valued fuzzy H-ideal of X if and only if for any $\tilde{t} \in D[0,1]$,
$\tilde{t} \geq \tilde{\mu}(0)$ the lower $\tilde{t}$-level cut $\tilde{\mu}^{\tilde{t}} = \{x\epsilon X, \tilde{\mu}(x) \leq \tilde{t}\}$ is a H-ideal of X

Proof.

Let $\tilde{\mu}$ be an anti- interval valued fuzzy H-ideal of X and let $\tilde{t}\epsilon D[0, 1]$ with $\tilde{\mu}(0) \leq \tilde{t}$ by (F_1), we have $\tilde{\mu}(0) \leq \tilde{\mu}(x)$ for all x∈X, but $\tilde{\mu}(x) \leq \tilde{t}$ for all $x \in \tilde{\mu}^{\tilde{t}}$ and so $0 \in \tilde{\mu}^{\tilde{t}}$. Let $x, y, z \in X$ be such that $(x * (y * z)) \in \tilde{\mu}^{\tilde{t}}$ and y∈ $\tilde{\mu}^{\tilde{t}}$ then $\tilde{\mu}((x * (y * z))) \leq \tilde{t}$ and $\tilde{\mu}(y) \leq \tilde{t}$ since $\tilde{\mu}$ is an anti- interval valued fuzzy H-ideal, it follow that $\tilde{\mu}(x * z) \leq rmax\{\tilde{\mu}((x * (y * z))), \tilde{\mu}(y)\} \leq \tilde{t}$ and hence $x * z \in \tilde{\mu}^{\tilde{t}}$, therefore $\tilde{\mu}^{\tilde{t}}$ is an H-ideal. Conversely, suppose that for each $\tilde{t}\epsilon D[0,1], \tilde{\mu}^{\tilde{t}}$ is a H-ideal of X, we only need to show that (F_1) , (F_3) are true. If (F_1) is does not hold then there exist $x_1 \epsilon X$ such that $\tilde{\mu}(0) \geq \tilde{\mu}(x_1)$ if we take $\tilde{t}_1 = \frac{1}{2}[\tilde{\mu}(0) + \tilde{\mu}(x_1)]$,then $\tilde{\mu}(0) > \tilde{t}_1$ and $\tilde{0} \leq \tilde{\mu}(x_1) < \tilde{t}_1 \leq \tilde{1}$ hence $x_1 \in \tilde{\mu}^{\tilde{t}_1}$ and $\tilde{\mu}^{\tilde{t}_1} \neq \varphi$. But $\tilde{\mu}^{\tilde{t}_1}$ is a H-ideal of X ,we have $0 \in \tilde{\mu}^{\tilde{t}_1}$ and so $\tilde{\mu}(0) \leq \tilde{t}_1$ contradiction .Hence $\tilde{\mu}(0) \leq \tilde{\mu}(x)$ for all $x \in X$,now prove that $\tilde{\mu}$ satisfies (F_3),if not ,then there exists x_0,y_0,z_0 in X such that $\tilde{\mu}_{\tilde{A}}(x_0 * z_0) > \tilde{\mu}_{\tilde{A}}(x_0 * (y_0 * z_0)) \vee \tilde{\mu}_{\tilde{A}}(y_0)$ Put $\tilde{t}_1 = 1/2[\tilde{\mu}_{\tilde{A}}(x_0 * z_0) + \tilde{\mu}_{\tilde{A}}(x_0 * (y_0 * z_0)) \vee \tilde{\mu}_{\tilde{A}}(y_0)]$ $then$ $\tilde{\mu}_{\tilde{A}}(x_0 * z_0) > \tilde{t}_1$ and $\tilde{0} \leq \tilde{t}_1 > \tilde{\mu}_{\tilde{A}}(x_0 * (y_0 * z_0)) \vee \tilde{\mu}_{\tilde{A}}(y_0) \leq \tilde{1}$, which gives $\tilde{\mu}_{\tilde{A}}(x_0 * (y_0 * z_0)) < \tilde{t}_1 and\ \tilde{\mu}(y_0) < \tilde{t}_1$ therefore $x_0 * (y_0 * z_0) \in \tilde{\mu}^{\tilde{t}_1}$, $y_0 \in \tilde{\mu}^{\tilde{t}_1}$ and hence $\tilde{\mu}_{\tilde{A}}(x_0 * z_0) \leq \tilde{t}_1$, a contradiction. Hence, the supposition is wrong this completes the proof**.**

Theorem.

An interval valued fuzzy set $\tilde{\mu}$ of a BCI-algebra X is an interval valued fuzzy H- ideal of X if and only if its complement $\tilde{\mu}^c$is an anti- interval valued fuzzy H-ideal of X.

Proof.

Let $\tilde{\mu}$ be a fuzzy H- ideal of a BCI-algebra X, and let $x, y, z \in X$, then $\tilde{\mu}^c(0) = \tilde{1} - \tilde{\mu}(0) \leq \tilde{1} - \tilde{\mu}(x) = \tilde{\mu}^c(x)$, and $\tilde{\mu}^c(x * z) = \tilde{1} - \tilde{\mu}(x * z) \leq \tilde{1} - rmin[\tilde{\mu}(x * (y * z)), \tilde{\mu}(y)] = \tilde{1} - rmin[\tilde{1} - \tilde{\mu}^c(x * (y * z)), \tilde{1} - \tilde{\mu}^c(y)] = rmax\{\tilde{\mu}^c(x * (y * z)), \tilde{\mu}^c(y)\}$, hence,$\tilde{\mu}^c$ is an anti- interval valued fuzz H-ideal of X. Now let $\tilde{\mu}^c$ be an anti- interval valued fuzzy H-ideal of X, and let$x, y, z \in X$, then $\tilde{\mu}(0) = \tilde{1} - \tilde{\mu}^c(0) \geq \tilde{1} - \tilde{\mu}^c(x) = \tilde{\mu}(x)$, and $\tilde{\mu}(x * z) = \tilde{1} - \mu^c(x * z) \geq \tilde{1} - rmax\{\mu^c(x * (y * z)), \tilde{\mu}^c(y)\} = \tilde{1} - rmax[\tilde{1} - \tilde{\mu}(x * (y * z)), \tilde{1} - \tilde{\mu}(y)] = rmin\{\tilde{\mu}(x * (y * z)), \tilde{\mu}(y)\}$ .hence, $\tilde{\mu}$ is an interval valued fuzzy H- ideal of X.

Theorem

Let $\tilde{\mu}$ be an anti- interval valued fuzzy ideal of BCI-algebra X then the following are equivalent:
i) $\tilde{\mu}$ is an anti- interval valued fuzzy H-ideal of X ((ii) $\tilde{\mu}((x * y) * z) \leq \tilde{\mu}(x * (y * z))$ $for\ all\ x, y, z \in X$
(iii)$\tilde{\mu}(x * y) \leq \tilde{\mu}(x * (0 * y))$

Proof. $(i) \longrightarrow (ii)$

Since µ̃ is an anti- interval valued fuzzy H-ideal of X, we have $\tilde{\mu}((x*y)*z) \leq rmax\{\tilde{\mu}((x*y)*(0*z)), \tilde{\mu}(0)\} = \tilde{\mu}((x*y)*(0*z))$ on the other hand $(x*y)*(0*z) = (x*y)*((y*z)*y) \leq x*(y*z)$,thus $\tilde{\mu}(x*(y*z)) \geq \tilde{\mu}((x*y)*(0*z)$

(ii)→(iii) letting y=0 and z=y in *(ii)*

(iii) → (i) since $(x*(0*y))*(x*(z*y)) \leq (z*y)*(0*y) \leq z$, we have µ̃ hypothesis $\tilde{\mu}(x*y) \leq rmax\{\tilde{\mu}((x*(z*y)), \tilde{\mu}(z)\}$. Therefore µ̃ is an anti- interval valued fuzzy H-ideal of X

Theorem .

Let µ̃ be an anti- interval valued fuzzy ideal of BCI-algebra if $\tilde{\mu}(x*y) \leq \tilde{\mu}(x)$ for all x,y∈X then µ̃ is an anti –interval valued fuzzy H-ideal of X

Proof.

Since µ̃ is an ideal of X, by hypothesis we have $rmax\,\{\tilde{\mu}(x*(y*z), \tilde{\mu}(y)\} \geq rmax\{\tilde{\mu}((x*z)*(y*z)), \tilde{\mu}(y*z)\} \geq \tilde{\mu}(x*z)\}$,forall *x,y,z∈X*

Theore.

An anti- interval valued fuzzy H-ideal µ̃ of BCI-algebra X is an anti- interval valued fuzzy subalgebra of X

Proof.

If µ̃ is an anti- interval valued fuzzy H-ideal ,then by(F_3) we have $\tilde{\mu}(x*z) \leq rmax\{\tilde{\mu}(x*(y*z), \tilde{\mu}(y)\}$ putting z=y Then $\tilde{\mu}(x*y) \leq rmax\{\tilde{\mu}(x), \tilde{\mu}(y)\}$ this shows that µ̃ is an anti- interval valued fuzzy subalgebra of X

Theorem .

Let µ̃ and ṽ be anti- interval valued fuzzy ideal of a BCI-algebra X such that µ̃ ≥ ṽ and $\tilde{\mu}(0) = \tilde{v}(0)$ if µ̃ is an anti- interval valued fuzzy H-ideal of X then so is $\tilde{v}$

Proof.

(iii) it is enough to show that $\tilde{v}(x*y) \leq \tilde{v}(x*(0*y))$ for each $x, y \in X$ putting $s = x*(0*y)$ we have $(x*s)*(0*y) = 0$ hence $\tilde{\mu}((x*s)*(0*y)) = \tilde{\mu}(0) = \tilde{v}(0)$.by theorem 3.8*(iii)* ,since µ̃ is a anti- interval valued fuzzy H-ideal of X ,$\tilde{\mu}((x*s)*y) \leq \tilde{\mu}((x*s)*(0*y)) = \tilde{v}(0)$ thus $\tilde{v}((x*y)*s) \leq \tilde{\mu}((x*y)*s) \leq \tilde{v}(0) \leq \tilde{v}(s)$ since ṽ is a anti- interval valued fuzzy ideal ,we have $\tilde{v}(x*y) \leq rmax\{\tilde{v}((x*y)*s), \tilde{v}(s)\} = \tilde{v}(s) = \tilde{v}\big(x*(0*y)\big)$

Homomorphism of anti- interval valued fuzzy H-ideal of BCI-algebra

Definiti .

Let (X , * , 0) and (Y , *`, 0`) be a BCI-algebras. A mapping f : X⟶Y is said to be a homomorphism if f (x * y) = f (x) *` f (y) for all x , y ∈ X.

***Definitio**:*

Let X and Y be two BCI-algebras, $\tilde{\mu}$ an interval valued fuzzy subset of X,$\tilde{\beta}$ an interval valued fuzzy subset of Y and f: X⟶ Y a BCI-homomorphism. The image of $\tilde{\mu}$ under f denoted by f ($\tilde{\mu}$) is an interval valued fuzzy set of Y defined by

$$f(\tilde{\mu})(y) = \begin{cases} \underset{x \in f^{-1}(y)}{sub}\, \tilde{\mu}(x) & if\ f^{-1}(y) \neq \emptyset \\ [0,0] & otherwise \end{cases}$$

the pre-image of $\tilde{\beta}$ under f denoted by $f^{-1}(\tilde{\beta})$ is an interval valued fuzzy set of X defined by :for all x∈ X, $f^{-1}(\tilde{\beta})(x) = \tilde{\beta}(f(x))$

Theorem.

Let f:x→Y be an onto BCI-homomorphism if an interval valued fuzzy subset $\tilde{B}$ of Y with membership function $\tilde{\mu}_{\tilde{B}}$ is an anti- interval valued fuzzy H-ideal ,then the interval valued fuzzy subset $f^{-1}(\tilde{B})$ of X with membership function $\tilde{\mu}_{f^{-1}}(\tilde{B})$ is also an anti- interval valued fuzzy H-ideal of X

Proof *.*

Let y∈Y since f is onto there exists x∈X ,y=f(x) since $\widetilde{B}$ is a anti- interval valued fuzzy H-ideal of Y .it follows that $\tilde{\mu}_{\widetilde{B}}(0) \leq \tilde{\mu}_{\widetilde{B}}(y)$ or $\tilde{\mu}_{\widetilde{B}}f(0) \leq \tilde{\mu}_{\widetilde{B}}f(x)$ then by definition $\tilde{\mu}_{f^{-1}(\widetilde{B})}(0) \leq \tilde{\mu}_{f^{-1}(\widetilde{B})}(x)$ for all x∈X next ,$\widetilde{B}$ is an anti- interval valued fuzzy H-ideal ,therefore for any y_1,y_2,y_3 in Y,$\tilde{\mu}_{\widetilde{B}}(y_1 * y_3) \leq \tilde{\mu}_{\widetilde{B}}(y_1 * (y_2 * y_3)) \vee \tilde{\mu}_{\widetilde{B}}(y_2)$ or $\tilde{\mu}_{\widetilde{B}}\, f(x_1 * x_3) \leq \tilde{\mu}_{\widetilde{B}}(f(x_1 * (x_2 * x_3)) \vee \tilde{\mu}_{\widetilde{B}}f(x_2))$ gives $\tilde{\mu}_{f^{-1}(\widetilde{B})}(x_1 * x_3) \leq \tilde{\mu}_{f^{-1}(\widetilde{B})}(x_1 * (x_2 * x_3)) \vee \mu_{f^{-1}(\widetilde{B})}(x_2)$, which proves that $f^{-1}(\widetilde{B})$ is a anti- interval valued fuzzy H-ideal of X this completes the proof

Theorem

Let f:X→Y be an onto BCI-homomorphism if a inf interval valued fuzzy subset $\tilde{A}$ of X with membership function $\tilde{\mu}_{\tilde{A}}$ is an anti-interval valued fuzzy H-ideal ,then the interval valued fuzzy subset f($\tilde{A}$) with membership function $\tilde{\mu}_{f(\tilde{A})}$ is also an anti-interval valued fuzzy H-ideal of Y

Proof :

Since interval valued fuzzy subset $\tilde{A}$ is an anti-interval valued fuzzy H-ideal of X therefore ,$\tilde{\mu}_{\widetilde{A}}(0) \leq \tilde{\mu}_{\widetilde{A}}(x)$ for all x∈X .since $0=f^{-1}(0)$ therefore,we have $\tilde{\mu}_{f(\widetilde{A})}(0)=\inf\tilde{\mu}_{\widetilde{A}}(x)=\tilde{\mu}_{\widetilde{A}}(0) \leq \tilde{\mu}_{\widetilde{A}}(x)$,x∈ $f^{-1}(0)$ or $\tilde{\mu}_{f(\widetilde{A})}(0) \leq \tilde{\mu}_{\widetilde{A}}(x)$ for all x∈X which implies $\tilde{\mu}_{f(\widetilde{A})}(0) \leq \inf\tilde{\mu}_{\widetilde{A}}(x)=\tilde{\mu}_{f(\widetilde{A})}(y)$,x∈$f^{-1}(y)$ or $\tilde{\mu}_{f(\widetilde{A})}(0) \leq \tilde{\mu}_{f(\widetilde{A})}(y)$ for all y in X next, let y_1,y_2,y_3 ∈Y and $x_1 \in f^{-1}(y_1)$, $x_2 \in f^{-1}(y_2)$, $x_3 \in f^{-1}(y_3)$ such that $\tilde{\mu}_{\widetilde{A}}(x_1*x_3)=\inf\tilde{\mu}_{\widetilde{A}}(t)$ where t∈ $f^{-1}(y_1*y_3)$, $\tilde{\mu}_{\widetilde{A}}(x_2)=\inf(\tilde{\mu}_{\widetilde{A}}(t),\tilde{\mu}_{\widetilde{A}}(x_1*(x_2*x_3))=\inf\tilde{\mu}_{\widetilde{A}}(t)$ where t∈ $f^{-1}(y_1*(y_2*y_3))$.By (ii)we have $\tilde{\mu}_{f(\widetilde{A})}(y_1*y_3)=\inf\tilde{\mu}_{\widetilde{A}}(t)$ (where t∈ $f^{-1}(y_1 * y_2)=\tilde{\mu}_{\widetilde{A}}(x_1*x_2) \leq \mu_{\widetilde{A}}(x_1*(x_2*x_3)) \vee \tilde{\mu}_{\widetilde{A}}(x_2)=\inf\tilde{\mu}_{\widetilde{A}}(t)$ (where t∈ $f^{-1}(y_1*(y_2*y_3)) \vee \inf\tilde{\mu}_{\widetilde{A}}(t)$ (where t∈ $f^{-1}(y_2)=\tilde{\mu}_{f(\widetilde{A})}(y_1*(y_2*y_3)) \vee \tilde{\mu}_{f(\widetilde{A})}(y_2)$ or $\tilde{\mu}_{f(\widetilde{A})}(y_1*y_3) \leq \tilde{\mu}_{f(\widetilde{A})}(y_1*(y_2*y_3)) \vee \tilde{\mu}_{f(\widetilde{A})}(y_2)$ this proves that f($\tilde{A}$) is also an anti –interval valued fuzzy H-ideal of Y this completes the proof

Cartesian product of anti- interval valued fuzzy H-ideals

Definition.

An interval valued fuzzy relation on any set X is an interval valued fuzzy subset $\tilde{\mu}$: X × X ⟶ D [0, 1].

Definition.

If $\tilde{\mu}$ is an interval valued fuzzy relation on a set X and $\tilde{\beta}$ is an interval valued fuzzy subset of X, then $\tilde{\mu}$ is an anti- interval valued fuzzy relation on $\tilde{\beta}$ if $\tilde{\mu}(x , y) \geq r\max\{\tilde{\beta}(x) , \tilde{\beta}(y)\}$ for all x , y ∈ X

Definition.

Let $\widetilde{A}$ and $\widetilde{B}$ be interval valued fuzzy subsets of a set X with membership function $\tilde{\mu}_{\widetilde{A}}$ and $\tilde{\mu}_{\widetilde{B}}$,respectively .then the Cartesian product $\widetilde{A} \times \widetilde{B}$ of $\widetilde{A}$ and $\widetilde{B}$ is an anti- interval valued fuzzy relation on a set X whose membership function $\tilde{\mu}_{\widetilde{A}X\widetilde{B}}$ is defined as $\tilde{\mu}_{\widetilde{A}X\widetilde{B}}(x, y) = \tilde{\mu}_{\widetilde{A}}(x) \vee \tilde{\mu}_{\widetilde{B}}(y)$ for all x, y ∈ X

Remark *.*

Let X and Y be BCI-algebras, we define * on X × Y by, for every *(x , y)* , *(u , v)* ∈ X × Y, *(x , y) * (u , v) = (x * u , y * v)*. Then clearly (X × Y ; * , (0 , 0)) is a BCI-algebra

Theore.

Let $\widetilde{A}$ and $\widetilde{B}$ be $\widetilde{A}$anti- interval valued fuzzy H-ideal of a BCI-algebra X with membership functions $\tilde{\mu}_{\widetilde{A}}$ and $\tilde{\mu}_{\widetilde{B}}$ respectively, then $\widetilde{A} \times \widetilde{B}$ is anti- interval valued fuzzy H-ideal of a BCI-algebra X×X with membership function $\tilde{\mu}_{\widetilde{A}X\widetilde{B}}$

Proof.

Let $(x, y) \in X \times X$ .then by definition $\tilde{\mu}_{\tilde{A}X\tilde{B}}(0,0) = \tilde{\mu}_{\tilde{A}}(0) \vee \tilde{\mu}_{\tilde{B}}(0) \leq \tilde{\mu}_{\tilde{A}}(x) \vee \tilde{\mu}_{\tilde{B}}(y) = \tilde{\mu}_{\tilde{A}X\tilde{B}}(x, y)$ *or* $\tilde{\mu}_{\tilde{A}X\tilde{B}}(0,0) \leq \tilde{\mu}_{\tilde{A}X\tilde{B}}(x, y)$ *for all* $(x, y) \in X \times X$. *next* ,consider $\tilde{\mu}_{\tilde{A}X\tilde{B}}((x_1, x_2) * ((y_1, y_2) * (z_1, z_2))) \vee \tilde{\mu}_{\tilde{A}\times\tilde{B}}(y_1, y_2) = \tilde{\mu}_{\tilde{A}\times\tilde{B}}((x_1, x_2) * (y_1 * z_2, y_1 * z_2)) \vee \tilde{\mu}_{\tilde{A}\times\tilde{B}}(y_1, y_2) = \tilde{\mu}_{\tilde{A}\times\tilde{B}}(x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) \vee \tilde{\mu}_{\tilde{A}\times\tilde{B}}(y_1, y_2) = [\tilde{\mu}_{\tilde{A}}(x_1 * (y_1 * z_1)) \vee \tilde{\mu}_{\tilde{B}}(x_2 * (y_2 * z_2))] \vee [\tilde{\mu}_{\tilde{A}}(y_1) \vee \tilde{\mu}_{\tilde{B}}(y_2)] = [\tilde{\mu}_{\tilde{A}}(x_1 * (y_1 * z_1)) \vee \tilde{\mu}_{\tilde{A}}(y_1)] \vee [\tilde{\mu}_{\tilde{B}}(x_2 * (y_2 * z_2) \vee \tilde{\mu}_{\tilde{B}}(y_2)] \geq \tilde{\mu}_{\tilde{A}}(x_1 * z_1) \vee \tilde{\mu}_{\tilde{B}}(x_2 * z_2) = \tilde{\mu}_{\tilde{A}\times\tilde{B}}(x_1 * z_1, x_2 * z_2) = \tilde{\mu}_{\tilde{A}\times\tilde{B}}((x_1, x_2), (z_1, z_2))$.

Theorem.

Let $\widetilde{A}$ and $\widetilde{B}$ be interval valued fuzzy subsets of a BCI-algebra X with membership functions $\tilde{\mu}_{\widetilde{A}}$ and$\tilde{\mu}_{\widetilde{B}}$, respectively. If $\widetilde{A} \times \widetilde{B}$ is a anti- interval valued fuzzy H-ideal of X×X with membership function $\tilde{\mu}_{\widetilde{A}X\widetilde{B}}$,then

(i)$\tilde{\mu}_{\widetilde{A}}(0) \leq \tilde{\mu}_{\widetilde{A}}(x)$ or $\tilde{\mu}_{\widetilde{B}}(0) \leq \tilde{\mu}_{\widetilde{B}}(x)$ for all x ∈ X

(ii)$\tilde{\mu}_{\tilde{B}}(0) \leq \tilde{\mu}_{\tilde{A}}(x)$ or $\tilde{\mu}_{\tilde{B}}(0) \leq \tilde{\mu}_{\tilde{B}}(x)$ for all $x \in X$
(iii)$\tilde{\mu}_{\tilde{A}}(0) \leq \tilde{\mu}_{\tilde{A}}(x)$ or $\tilde{\mu}_{\tilde{A}}(0) \leq \tilde{\mu}_{\tilde{B}}(x)$ for all $x \in X$
(iv)$\tilde{A}$ or $\tilde{B}$ is a anti- interval valued fuzzy H-ideal of X

Proof. *(i)*

Let $\tilde{\mu}_{\tilde{A}}(0) > \tilde{\mu}_{\tilde{A}}(x)$ and $\tilde{\mu}_{\tilde{B}}(0) > \tilde{\mu}_{\tilde{B}}(y)$ for some x,y in X .then $\tilde{\mu}_{\tilde{A}X\tilde{B}}(x,y) = \tilde{\mu}_{\tilde{A}}(x) \vee \tilde{\mu}_{\tilde{B}}(y) < \tilde{\mu}_{\tilde{A}}(0) \vee \tilde{\mu}_{\tilde{B}}(0) = \tilde{\mu}_{\tilde{A}X\tilde{B}}(0,0)$ or $\tilde{\mu}_{\tilde{A}X\tilde{B}}(x,y) < \tilde{\mu}_{\tilde{A}X\tilde{B}}(0,0)$ for all $(x,y) \in X \times X$,a contradiction. This proves (i)
(ii) Let $\tilde{\mu}_{\tilde{B}}(0) > \tilde{\mu}_{\tilde{A}}(x)$ and $\tilde{\mu}_{\tilde{B}}(0) > \tilde{\mu}_{\tilde{B}}(y)$ for some x,y in X .then $\tilde{\mu}_{\tilde{A}X\tilde{B}}(0,0) = \tilde{\mu}_{\tilde{A}}(0) \vee \tilde{\mu}_{\tilde{B}}(0) = \tilde{\mu}_{\tilde{B}}(0)$ it follows that $\tilde{\mu}_{\tilde{A}X\tilde{B}}(x,y) = \tilde{\mu}_{\tilde{A}}(x) \vee \tilde{\mu}_{\tilde{B}}(y) < \tilde{\mu}_{\tilde{B}}(0) \vee \tilde{\mu}_{\tilde{B}}(0) = \tilde{\mu}_{\tilde{B}}(0) = \tilde{\mu}_{\tilde{A}X\tilde{B}}(0,0)$ or $\tilde{\mu}_{\tilde{A}X\tilde{B}}(x,y) < \tilde{\mu}_{\tilde{A}X\tilde{B}}(0,0)$, a contradiction. This proves (ii)
(iii)similar to (ii)
(iv)By (i),let $\tilde{\mu}_{\tilde{B}}(0) \leq \tilde{\mu}_{\tilde{B}}(x)$ for all $x \in X$. From (iii),take $\tilde{\mu}_{\tilde{B}}(x) \geq \tilde{\mu}_{\tilde{A}}(0)$ for all $x \in X$, then $\tilde{\mu}_{\tilde{A}X\tilde{B}}(0,x) = \tilde{\mu}_{\tilde{A}}(0) \vee \tilde{\mu}_{\tilde{B}}(x) = \tilde{\mu}_{\tilde{B}}(x)$ (1)
Since $\tilde{A} \times \tilde{B}$ is a anti- interval valued fuzzy H-ideal, we have $\tilde{\mu}_{\tilde{A}X\tilde{B}}((x_1,x_2) * (z_1,z_2)) \leq \tilde{\mu}_{\tilde{A}X\tilde{B}}((x_1,x_2) * ((y_1,y_2) * (z_1,z_2))) \vee \tilde{\mu}_{\tilde{A}X\tilde{B}}(y_1,y_2)$ or $\tilde{\mu}_{\tilde{A}X\tilde{B}}(x_1 * z_1, x_2 * z_2) \leq \tilde{\mu}_{\tilde{A}X\tilde{B}}(x_1 * (y_1 * z_1), (x_2 * (y_2 * z_2)) \vee \tilde{\mu}_{\tilde{A}X\tilde{B}}(y_1,y_2)$. If $x_1 = y_1 = z_1 = 0$, then $\tilde{\mu}_{\tilde{A}X\tilde{B}}(0, x_2 * z_2) \leq \tilde{\mu}_{\tilde{A}X\tilde{B}}(0, x_2 * (y_2 * z_2)) \vee \tilde{\mu}_{\tilde{A}X\tilde{B}}(0,y_2)$. By (i), we have
$\tilde{\mu}_{\tilde{B}}(x_2 * z_2) \leq \tilde{\mu}_{\tilde{B}}(x_2 * (y_2 * z_2)) \vee \tilde{\mu}_{\tilde{B}}(y_2)$.this proves that $\tilde{B}$ is an anti- interval valued fuzzy H-ideal of X .similar is the case when $\tilde{\mu}_{\tilde{A}}(x) \geq \tilde{\mu}_{\tilde{A}}(0)$,for all x∊X and $\tilde{\mu}_{\tilde{A}}(x) \geq \mu_B(0)$ for all $x \in X$.this gives $\tilde{A}$ is a anti- interval valued fuzzy H-ideal of X .this completes the proof .

Definition .

Let $\tilde{B}$ be an interval valued fuzzy subset of a BCI-algebra X. then the strongest anti- interval valued fuzzy relation on X that is an anti- interval valued fuzzy relation $\tilde{A}$ on $\tilde{B}$ is $\tilde{A}_{\tilde{B}}$ whose membership function is given by
$\tilde{\mu}_{\tilde{A}_{\tilde{B}}}(x,y)=\tilde{\mu}_{\tilde{B}}(x) \vee \tilde{\mu}_{\tilde{B}}(y)$ for all x, y in X

Theorem.

Suppose $\tilde{B}$ be an interval valued fuzzy subset of a BCI-algebra X with membership function $\tilde{\mu}_{\tilde{B}}$ and $\tilde{A}_{\tilde{B}}$ the strongest anti- interval valued fuzzy relation on X with membership function $\tilde{\mu}_{\tilde{A}_{\tilde{B}}}$ then $\tilde{B}$ is an anti- interval valued fuzzy H-ideal of X if and only if $\tilde{A}_{\tilde{B}}$ is an anti- interval valued fuzzy H-ideal of X×X

Proof.

Assume that $\tilde{B}$ is an anti- interval valued fuzzy H-ideal of X .then $\tilde{\mu}_{\tilde{A}_{\tilde{B}}}(0,0) = \tilde{\mu}_{\tilde{B}}(0) \vee \tilde{\mu}_{\tilde{B}}(0) \leq \tilde{\mu}_{\tilde{B}}(x) \vee \tilde{\mu}_{\tilde{B}}(y) = \tilde{\mu}_{\tilde{A}_{\tilde{B}}}(x,y)$ for all (x,y) in $X \times X$. next $\tilde{\mu}_{\tilde{A}_{\tilde{B}}}((x_1,x_2) * (z_1,z_2)) = \tilde{\mu}_{\tilde{A}_{\tilde{B}}}(x_1 * z_1, x_2 * z_2) = \tilde{\mu}_{\tilde{B}}(x_1 * z_1) \vee \tilde{\mu}_{\tilde{B}}(x_2 * z_2) \leq [\tilde{\mu}_{\tilde{B}}(x_1 * (y_1 * z_1)) \vee \tilde{\mu}_{\tilde{B}}(y_1)] \vee \tilde{\mu}_{\tilde{B}}(x_2 * (y_2 * z_2)) \vee \tilde{\mu}_{\tilde{B}}(y_2)] = [\tilde{\mu}_{\tilde{B}}(x_1 * (y_1 * z_1)) \vee \tilde{\mu}_{\tilde{B}}(x_2 * (y_2 * z_2))] \vee [\tilde{\mu}_{\tilde{B}}(y_1) \vee \tilde{\mu}_{\tilde{B}}(y_2)] = \tilde{\mu}_{\tilde{A}_{\tilde{B}}}(x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) \vee$
$\tilde{\mu}_{\tilde{A}_{\tilde{B}}}(y_1,y_2)$ for all $(x_1,x_2),(y_1,y_2),(z_1,z_2)$ in X×X.

this proves that $\tilde{A}_{\tilde{B}}$ is a anti- interval valued fuzzy H-ideal of X×X.conversely ,suppose $\tilde{A}_{\tilde{B}}$ be an anti- interval valued fuzzy H-ideal of X×X then for all $(x,x) \in X \times X$.$\tilde{\mu}_{\tilde{B}}(0) \vee \tilde{\mu}_{\tilde{B}}(0) = \tilde{\mu}_{\tilde{A}_{\tilde{B}}}(0,0) < \tilde{\mu}_{\tilde{A}_{\tilde{B}}}(x,x) = \tilde{\mu}_{\tilde{B}}(x) \vee \tilde{\mu}_{\tilde{B}}(x)$ or $\tilde{\mu}_{\tilde{B}}(0) < \tilde{\mu}_{\tilde{B}}(x)$ for all x∊X. next , let $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in X \times X$, then $\tilde{\mu}_{\tilde{B}}(x_1 * z_1) \vee \tilde{\mu}_{\tilde{B}}(x_2 * z_2) = \tilde{\mu}_{\tilde{A}_{\tilde{B}}}(x_1 * z_1, x_2 * z_2) = \tilde{\mu}_{\tilde{A}_{\tilde{B}}}((x_1,x_2),(z_1,z_2)) \leq \tilde{\mu}_{\tilde{A}_{\tilde{B}}}((x_1,x_2) * ((y_1,y_2) * (z_1,z_2))) \vee \tilde{\mu}_{\tilde{A}_{\tilde{B}}}(y_1,y_2) = \tilde{\mu}_{\tilde{A}_{\tilde{B}}}(x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) \vee \tilde{\mu}_{\tilde{A}_{\tilde{B}}}(y_1,y_2) = [\tilde{\mu}_{\tilde{B}}(x_1 * (y_1 * z_1) \vee \tilde{\mu}_{\tilde{B}}(x_2 * (y_2 * z_2))] \vee [\tilde{\mu}_{\tilde{B}}(y_1) \vee \tilde{\mu}_{\tilde{B}}(y_2)] = [\tilde{\mu}_{\tilde{B}}(x_1 * (y_1 * z_1)) \vee \tilde{\mu}_{\tilde{B}}(y_1)] \vee [\tilde{\mu}_{\tilde{B}}(x_2 * (y_2 * z_2)) \vee \tilde{\mu}_{\tilde{B}}(y_2)]$ we take $x_2 = y_2 = z_2 = 0$, we get $\tilde{\mu}_{\tilde{B}}(x_1 * z_1) \vee \tilde{\mu}_{\tilde{B}}(0) \leq \tilde{\mu}_{\tilde{B}}(x_1 * (y_1 * z_1)) \vee \tilde{\mu}_{\tilde{B}}(y_1) \vee \tilde{\mu}_{\tilde{B}}(0)$

Or $\quad \tilde{\mu}_{\tilde{B}}(x_1 * z_1) \leq \tilde{\mu}_{\tilde{B}}(x_1 * (y_1 * z_1)) \vee \tilde{\mu}_{\tilde{B}}(y_1)$

Whish proves that $\tilde{B}$ is an anti- interval valued fuzzy H-ideal of X

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